

MATH 2135. REVIEW FOR MIDTERM II

1. Read the following sections in the book: 2.1–2.3, 2.8–2.9, 3.1–3.3, 4.1–4.7. For Section 3.3 you only need to read the part about “Determinant as Area or Volume”.

2. Let A, B be $n \times n$ matrices.

(a) State the definition of the inverse of A ; that is, explain what it means for A to be invertible, and explain what the inverse of A is when A is invertible.

(b) Find the inverse of the matrix

$$C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

(c) Recall that $\det(AB) = \det A \det B$. Use this fact (which you should know) and Part (a) to prove that $\det(A^{-1}) = 1/\det A$.

(d) Recall that $\det A = \det A^T$, where A^T stands for the transpose of A . Use this fact (which you should know) and the previous parts to compute $\det(C^2(C^T)^3C^{-1})$ for the matrix C from Part (b).

3. Let A be an $n \times n$ matrix and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the map defined by $T(x) = Ax$ for any $x \in \mathbb{R}^n$. Fill in the blanks below for the characterization of invertible matrices.

Theorem 1. *The following are equivalent.*

(a) *A is invertible.*

(b) *An echelon form of A contains ... pivot columns.*

(c) *An echelon form of A contains ... zero rows.*

(d) *The columns of A are*

(e) *The columns of A span*

(f) *The rank of A is*

(g) *The nullity of A is*

(h) *The map T is injective.*

(i) *The map T is*

- (j) The map T is bijective.
- (k) The kernel of T is
- (l) The image of T is
- (m) The equation $Ax = 0$ has . . . solution.
- (n) The equation $Ax = b$ has . . . solution for all $b \in \mathbb{R}^n$.
- (o) $\det A$. . . zero.
- (p) A^T is
4. (a) State the definitions of the null space, column space, and row space of a matrix.
- (b) Let A be an 4×7 matrix with 4 pivot columns. Find the dimensions of the null space, column space and row space of A , then do the same for the transpose A^T . Is $\text{Col } A = \mathbb{R}^4$? Is $\text{Null } A = \mathbb{R}^3$?
- (c) Let

$$B = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Find bases for $\text{Null } B$, $\text{Col } B$, $\text{Row } B$, then conclude what their dimensions are.

- (d) State the rank-nullity theorem for matrices.
5. Let
- $$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$
- (a) Compute $\det A$, then use it to compute $\det \left(\frac{1}{2}A \right)$.
- (b) Let B be the matrix obtained from A by interchanging the top and bottom row. What is $\det B$ and why?
- (c) Let C be the matrix obtained from B by adding 3 times the first row to the third row. What is $\det C$ and why?

6. Let V, W be vector spaces. and let $T : V \rightarrow W$ be a linear map.

- (a) State the definition of a subspace of V .
- (b) Prove that the image of T is a subspace of W .
- (c) Let w be a nonzero element in W . Prove that the set $\{v \in V : T(v) = w\}$ is not a subspace of V .
7. Let P_3 be the vector space of all polynomials in indeterminate t of degree at most 3.
- (a) Prove that the formal differentiation map $d : P_3 \rightarrow P_3$, i.e., the map defined by $d(f) = f'$ for any $f \in P_3$ (where we treat each polynomial f as a function $f = f(t)$ of t), is a linear map. Clearly point out any facts from calculus you are using.
- (b) Describe the kernel of d , then find a basis of the kernel.
- (c) Describe the image of d , then find a basis of the image.
- (d) Verify that the set $B = \{1, 2 + t, 3 + t + t^2, 4 + t^3\}$ is a basis of P_3 .
- (e) Let $g = t^3 + 2t^2 + t - 10 \in P_3$ and let B be as in Part (d). Find $[g]_B$.
8. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ where

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

- (a) Prove that both B and C are bases of \mathbb{R}^2 .
- (b) Find $\mathcal{P}_{C \leftarrow B}$, the change-of-basis matrix from B to C .
- (c) Use the previous part to find $\mathcal{P}_{B \leftarrow C}$.
- (d) Suppose $[v]_C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for some $v \in \mathbb{R}^2$. Find v and $[v]_B$.