

MATH 2135. REVIEW FOR CHAPTERS 5 & 6

1. Read the sections 5.1–5.4 and 6.1–6.3 in the book. For 6.2 you can ignore the discussion on orthonormal sets.
2. Let A be an $n \times n$ matrix.
 - (a) State the definition of an eigenvector and the corresponding eigenvalue for A . Can an eigenvector be the zero vector? At most how many eigenvalues can A have?
 - (b) Given a vector v , how should we check whether v is an eigenvector of A ? Given a scalar λ , how should we check whether λ is an eigenvalue of A ? Do Exercises 5 and 7 in Section 5.1.
 - (c) What is the characteristic polynomial of A ? How does it relate to the problem of finding eigenvalues of A , and what is the algebraic multiplicity of each eigenvalue? Do Exercise 15 of Section 5.2.
 - (d) For each eigenvalue λ of A , how many eigenvectors does the corresponding eigenspace E_λ (i.e., the set of eigenvectors of A with eigenvalue λ) have? How can we find a basis of E_λ ?
 - (e) Recall that for each eigenvalue λ of A , the dimension of the eigenspace E_λ is called the geometric multiplicity of A . What can we say about the relationship between the geometric multiplicity and the algebraic multiplicity of each eigenvalue?
 - (f) What does it mean for A to be diagonalizable? How does diagonalizability relate the algebraic and geometric multiplicities of the eigenvalues of A ? Read Examples 3, 4 and 6 in Section 5.3.
3. Let A be an $n \times n$ matrix?
 - (a) What does it mean to diagonalize A ? Describe how to test whether it can be done, and how to do it when possible.
 - (b) Do Exercises 1, 3, and 11 in Section 5.3.
4. Let V and W be abstract vector spaces. Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis of B , and let $C = \{c_1, c_2, \dots, c_m\}$ be a basis of C .

- (a) Given a vector $w \in W$, what is the coordinate vector $[w]_C$ of w with respect to C ? Which vector space is $[w]_C$ in?
- (b) Find $[w]_C$ for each of the following cases.

i. $W = \mathbb{R}^3$, $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and C is the standard basis

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

ii. $W = \mathbb{R}^3$, $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and C is the basis

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

- iii. $W = P_2$, the set of polynomials with real coefficient of degree at most 2, $w = 2t + 3t^2$, and C is the standard basis $\{1, t, t^2\}$.
- iv. $W = P_2$, the set of polynomials with real coefficient of degree at most 2, $w = 2t + 3t^2$, and C is the basis $\{t^2, t, 1\}$.
- v. $W = P_2$, the set of polynomials with real coefficient of degree at most 2, $w = 2t + 3t^2$, and C is the basis $\{2t, t^2 - 2, 2\}$.
- (c) Now let $T : V \rightarrow W$ be a linear map. How do we find the matrix $[T]_B^C$, the matrix of T relative to B and C ? In particular, how do we find the j -th column of $[T]_B^C$ for $1 \leq j \leq n$? Do Exercises 1, 2 in Section 5.4.
- (d) Find $[T]_B^C$ in the following cases, where T stands for the formal differentiation map.
- $V = P_2$, $B = \{1, t, t^2\}$, $W = P_2$, and C is as in (b.iii).
 - $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, and C is as in (b.iii).
 - $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, and C is as in (b.iv).

- iv. $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, and C is as in (b.v).
5. Let A be an $n \times n$ matrix, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear map given by $T(x) = Ax$ for all $x \in \mathbb{R}^n$. How does the diagonalizability of A relate to the problem of trying to find a basis B of \mathbb{R}^n for which $[T]_B^B$ is diagonal? Read Example 3 and do Exercises 13 and 15 in Section 5.4.
6. Let $B = \{b_1, b_2, \dots, b_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ be two bases of \mathbb{R}^n .
- (a) How do we find the change-of-basis matrix $\mathcal{P}_{C \leftarrow B}$ from B to C ? What equation should hold for $\mathcal{P}_{C \leftarrow B}$ and the coordinate vectors $[v]_B, [v]_C$ where v is an arbitrary vector in \mathbb{R}^n .
- (b) How are the two change-of-basis matrices $\mathcal{P}_{C \leftarrow B}$ and $\mathcal{P}_{B \leftarrow C}$ related?
7. Let u, v be vectors in \mathbb{R}^n .
- (a) State the definition of the inner product $u \cdot v$ of u and v .
- (b) Explain why $u \cdot u \geq 0$ for all $u \in \mathbb{R}^n$.
- (c) Prove that for each $u \in \mathbb{R}^n$, the map “dot with u ”, i.e., the map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $f(v) = u \cdot v$, is a linear map.
- (d) Define the length of u . Explain how to “normalize” any given vector u , i.e., how to find a unit vector with length 1 which points in the same direction as u . Do Exercise 9 in Section 6.1.
- (e) How is the distance between u and v defined? Do Exercise 13 in Section 6.1.
- (f) Explain, in terms of inner products, what it means for u and v to be orthogonal.
8. Let $B = \{u_1, u_2, \dots, u_n\}$ be an orthogonal basis of \mathbb{R}^n and let $y \in \mathbb{R}^n$.
- (a) Let $1 \leq j \leq n$. What’s the j -th coordinate in the vector $[y]_B$?
- (b) Write down the formula for the orthogonal projection \hat{y}_j of y onto u_j .
- (c) Let $1 \leq i < j \leq n$. Are the orthogonal projections \hat{y}_i and \hat{y}_j orthogonal? Why?
- (d) Let I be a subset of B and let W be the subspace of \mathbb{R}^n spanned by the elements in I . How do we find the orthogonal projection $\text{proj}_W y$ of y onto W , and how do we find the distance from y to W ? Read Examples 3, 4 and do Problems 11, 13, 15 in Section 6.3.