Math 2135. Solutions to Review for Midterm II

1. Do read them!

2. (a) Answer: A is invertible if there is another $n \times n$ matrix M such that $AM = I_n = MA$. When this is the case, any such matrix M is called an inverse of A (but there is at most one such matrix M, so the inverse of A is unique if it exists).

   (b) The inverse should be

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

   To get this, we should use the inversion algorithm: row-reduce the matrix $[A|I_3]$ to the point where the left half is I_3, whence the right half is the desired inverse.

   (c) Since $AA^{-1} = I_n$ and $\det I_n = 1$, we have

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

   It follows that $\det(A^{-1}) = 1/\det A$.

   (d) We have

$$\det(C^2(C^T)^3C^{-1}) = [\det(C)]^2[\det(C^T)]^3\det(C^{-1}) = [\det(C)]^2[\det(C)]^3[\det(C)]^{-1} = [\det(C)]^4,$$

   where the first two equalities follow from the facts mentioned in (c) and (d) (can you identity what facts are used in each step precisely?). A quick computation gives $\det(C) = -2$, therefore the desired determinant should be $(-2)^4 = 16$.

3. Here's the completed statement:

Theorem 1. *The following are equivalent.*

(a) A is invertible.

(b) An echelon form of A contains **n** *pivot columns.*

(c) An echelon form of A contains ***no*** *zero rows.*

(d) The columns of A are ***linearly independent***.

(e) *The columns of A span* $\mathbb{R}^{\mathbf{n}}$.

(f) *The rank of A is* $\mathbf{n}$.

(g) *The nullity of A is* $\mathbf{0}$.

(h) *The map T is injective.*

(i) *The map T is* ***surjective****.*

(j) *The map T is bijective.*

(k) *The kernel of T is* ***trivial, i.e.,*** $\{\mathbf{0}\}$.

(l) *The image of T is* **(*all of*)** $\mathbb{R}^{\mathbf{n}}$.

(m) *The equation $Ax = 0$ has* ***a unique/only the trivial*** *solution.*

(n) *The equation $Ax = b$ has* ***a unique*** *solution for all $b \in \mathbb{R}^n$.*

(o) $\det A$ ***is not*** *zero.*

(p) A^T *is* ***invertible****.*

4. (a) See Sections 4.2 and 4.5 for the definitions.

(b) For A, the dimensions of the column space, row space and null space are $4, 4$ and 3, respectively. For A^T, the row and column spaces have dimension 4 while the null space has dimension 0. (Why?)

Since each column of A is a vector in $\mathbb{R}^4$, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^4$. Since the column space has dimension $4 = \dim(\mathbb{R}^4)$, it follows that $\operatorname{Col} A = \mathbb{R}^4$. On the other hand, it is wrong to say that $\operatorname{Null} A = \mathbb{R}^3$, since the null space consists of vectors of length 7, not 3.

(c) Recall the correct algorithms for finding the bases: First we need to compute an echelon form C of B to see where the pivots are. The pivot columns in B itself then form a basis for the column space, and the pivot rows in the echelon form C form a basis for the row space. To get a basis for the null space we need to further reduce C to the reduced echelon form, solve the equation $Bx = 0$ in parametric vector form, then conclude that the constant vectors in the parametric form is a basis for $\operatorname{Null} B$. The reduced echelon form of B is

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and a basis for $\operatorname{Null} B$ is given by

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

I'll leave out the other details.

(d) The rank-nullity theorem states that the for every $m \times n$ matrix A, we have $\operatorname{Rank}(A) + \operatorname{Nullity}(A) = n$, i.e., the rank and nullity of A add up to the number of columns of A.

5. Let

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

(a) We can use a cofactor expansion to compute $\det A$; the answer should be -36. To obtain $\frac{1}{2}A$ from A we need to scale all four rows of A by $1/2$. Each time we scale a row by a number k the determinant is also scaled by k, therefore $\det(A/2) = (-36) \times (1/2)^4 = -36/16 = -9/4$.

(b) We have $\det B = -\det A = 36$, since interchanging two rows negates the determinant.

(c) We have $\det C = \det B = 36$, since adding a multiple of a row to another row does not change the determinant.

6. Let V, W be vector spaces and let $T : V \to W$ be a linear map.

(a) See Section 4.1.

(b) We should check that $\operatorname{Im} T$ satisfies the three properties in the definition of a subspace. We did this and many similar problems in class; see the relevant notes.

(c) One way to show this is to prove that the set is not closed under addition: if $v, v' \in T$, then $T(v + v') = T(v) + T(v') = w + w = 2w \neq w$, where the first equality follows from linearity of T and $2w \neq w$ since $w \neq 0$. It follows that the given set is a subspace of W. You can also argue that the set fails to be a subspace because it is not closed under scalar multiplication or that it doesn't contain zero (why?).

7. (a) The sum rule and scaling rule for differentiation from calculus state that $(f+g)' = f' + g'$ and $(cf)' = c \cdot f'$ for any $c \in \mathbb{R}$ and $f, g \in P_3$, which asserts exactly that d respects addition and scalar multiplication, therefore d is linear.

   (b) An arbitrary element $f \in P_3$ takes the form $f = a + bt + ct^2 + dt^3$, whence $d(f) = b + 2ct + 3dt^2$. Thus, $d(f) = 0$ if and only if $b = c = d = 0$, if and only if $f = a$, i.e., if and only if f is a constant polynomial. Note that the constant polynomial $1 \in P_3$ spans the set of all constant functions. Being a set with a single nonzero element, $\{1\}$ is also linearly independent, therefore $\{1\}$ forms a basis of the kernel.

   (c) By the generic form of f and $d(f)$ in (b), the image of d is the set $\{b+2ct+3dt^2 : b, c, d \in \mathbb{R}\} = \{B + Ct + Dt^2 : B, C, D \in \mathbb{R}\}$. The elements $1, t, t^2$ clearly span this set and are linear independent (why?), therefore $\{1, t, t^2\}$ is a basis for the image.

   (d) It suffices to show that the coordinate vectors of the vectors in B with respect to the basis $B' := \{1, t, t^2, t^3\}$ of P_3 forms a basis of $\mathbb{R}^4$. The coordinate vectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

   To check they are a basis of $\mathbb{R}^4$, it suffices to check that the matrix $A = [v_1|v_2|v_3|v_4]$ is invertible. Now A is in fact already in echelon form and has no zero row, so A is invertible, as desired.

   (e) The coordinate vector of g with respect to the basis B' in P_3 is

$$v = \begin{bmatrix} -10 \\ 1 \\ 2 \\ 1 \end{bmatrix},$$

therefore $[g]_B$ equals $[v]_{B'}$, so to find $[g(t)]_B$ it suffices to solve the equation $Ax = v$. The answer should be

$$[g]_B = \begin{bmatrix} -18 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

8. (a) If suffices to check that the matrices $M_B = [b_1|b_2]$ and $M_C = [c_1|c_2]$ are invertible. We can do so by computing their determinant and noting that they are not zero.

   (b) Recall that $\mathcal{P}_{C\leftarrow B}$ should be the matrix on the right half of the result we get when we row reduce $[M_C|M_B]$ to the point where the left half is I_2. In this case, we should get

$$\mathcal{P}_{C\leftarrow B} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}.$$

   (c) We have

$$\mathcal{P}_{B\leftarrow C} = (\mathcal{P}_{C\leftarrow B})^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}.$$

   (d) We should get

$$v = M_C[v]_C = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

and

$$[v]_B = \mathcal{P}_{B\leftarrow C}[v]_C = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$