Math 2135. Proof practice worksheet

Let V, W be arbitrary vector spaces. Fill in the blank spaces below to complete the proofs.

1. Let $T : V \to W$ be a linear map. Prove that $T(0_V) = 0_W$.

   Proof. We have
   $$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$
   where the first equality holds because ____
   and the second equality holds because ____.
   Let $w = T(0_V)$. Then the above equality becomes $w = w + w$. Adding ____________ to both sides of the new equality, we get ____.
   It follows that $w = 0$, and we are done. □

2. Let $T : V \to W$ be a linear map. Prove that $\ker T$ is closed under addition.

   Proof. We need to prove that if u, v are vectors in $\ker T$, then $u + v$ is also in $\ker T$.

   So, suppose $u, v \in \ker T$. Then …

   … So $u + v \in \ker T$, and we are done. □

3. Let $T : V \to W$ be a linear map. Prove that $\ker T$ is a subspace of V. Feel free to invoke the results of Problems 1 and 2.

Proof. We need to prove that $\ker T$ satisfies three properties:

(a) We need to show that ...

(b) We need to show that $\ker T$ is closed under addition.

(c) We need to show that ...

□

4. Prove that for every nonempty subset S of V, the span of S is a subspace of V.

Proof. We need to to show that Span(S) satisifies three properties . . .

□

5. Suppose that $B = \{v_1, \ldots, v_p\}$ is a basis of V. Then every element $v \in V$ can be written as a unique linear combination of $v_1, \ldots, v_p$.

   Proof. Let $v \in V$. We first note that v can be written as a linear combination of $v_1, \ldots, v_p$ because ____.
   It remains to prove uniqueness, that is, we need to prove that if $v = c_1c_1 + \cdots + c_pv_p$ and $v = d_1v_1 + \cdots + d_pv_p$ for scalars $c_1, \ldots, c_p$ and $d_1, \ldots, d_p$, then _________________________.
   So, suppose $v = c_1c_1 + \cdots + c_pv_p = d_1v_1 + \cdots + d_pv_p$. Then ...

   □