Math 2135. Review for Midterm I

1. Read Sections 1.1–1.5, 1.7–1.9, and 2.1–2.2 of the textbook. For Section 2.2 you can ignore the material after Example 4.

2. State the definition of linear maps, i.e., explain what it means for a map $T : V \to W$ between vector spaces V, W to be linear. Make sure you use the proper quantifiers like "for all" or "for some" where necessary.

3. Consider the following four matrices.

$$A = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 6 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) Determine which of the four matrices is/are in echelon form. For each matrix not in echelon form, point out one reason why it isn't.

   (b) Which of the matrices is/are in reduced echelon form? Explain.

   (c) Out of the four matrices, pick any that is not in the reduced echelon form, then compute its reduced echelon form and find the pivot column(s) of the matrix.

4. Consider the following system of linear equations.

$$\begin{cases} -2x + 5y + 5z - 5w &= 3 \\ x - 4y - z &= 1 \\ x - 2y &= 0 \end{cases}$$

   (a) Write the system as a vector equation.

   (b) Explain, without doing any computation, why the system must have infinitely many solutions.

5. Consider the matrix equation

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & 4 & 2 \\ 2 & -14 & 1 & 10 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ -3 \\ 7 \\ 7 \end{bmatrix}.$$

Solve the equation. Write the general solution in parametric vector form, then describe the solution set geometrically.

6. Determine whether each of the following maps is linear. Explain your reasoning.

$$\begin{aligned} &T_1 : \mathbb{R} \to \mathbb{R}, & T_1(x) &= 2x + 3 \\ &T_2 : \mathbb{R}^2 \to \mathbb{R}^2, & T_2(x, y) &= (x + y, y - 1) \\ &T_3 : \mathbb{R}^3 \to \mathbb{R}^3, & T_3(x, y, z) &= (x^2, y + 2z, x + y + z) \\ &T_4 : \mathbb{R}^4 \to \mathbb{R}^4, & T_4(x, y, z, w) &= (x + y, x, y + 2z, z - x - y) \end{aligned}$$

7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let A be the standard matrix of A. Fill in the blanks below.

Theorem 1. *The following are equivalent.*

(a) The map T is injective.

(b) The kernel of T equals ....

(c) The columns of A are ....

(d) Every echelon form of A satisfies the condition that ....

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which first reflects points through the x-axis and then reflects points through the line $y = x$.

(a) Find the matrix of T with respect to the standard basis.

(b) Determine if T is surjective.

(c) Determine if T is injective.

9. Consider the matrix

$$A = \begin{bmatrix} 3 & 5 & h \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) Find all value(s) of h for which the third column of A is in the span of the first two columns of A.

(b) Find all value(s) of h for which the columns of A are linearly independent.

(c) Find all value(s) of h for which the columns of A span $\mathbb{R}^3$.

(d) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. For which values of h is T surjective, and for which values of h is T injective? You may use the results from the previous parts.

10. (a) [**1pt**] Suppose A, B are matrices such that A is 5×3, AB is defined, and AB is 5×7. What is the size of B?

(b) [**4pts**] Let

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

Determine which of the products $AB, BA, A^T B^T, B^T A^T$ are defined, and compute all the products that are.

11. Find two 2×2 matrices A, B such that $AB \neq BA$.

12. Find three 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$.

13. Recall that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc \neq 0$ is invertible and has inverse

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Use this fact to solve the vector equation

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

(*Hint*: Rewrite the vector equation as a matrix equation first.)