Math 2135. Lecture 9.

09. 13. 2021.

Last time: · more criteria for lin ind/dep, and their proofs.

Let $S = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$. I'll often omit $\rightarrow$ on vectors.

of course, we also have the EF. criterion.

- if $0 \in S$, then S is lin. dep.
- if $|S| = 1$, then S is lin ind unless the unique elt in S is zero.
- if $|S| = 2$, then S is lin dep. iff one elt in S is a multiple of the other.
- if $k > n$ then S is lin dep. (if $k < n$ then S doesn't span $\mathbb{R}^n$).
- S is lin dep iff some elt in S is a lin comb. of the other elts of S.

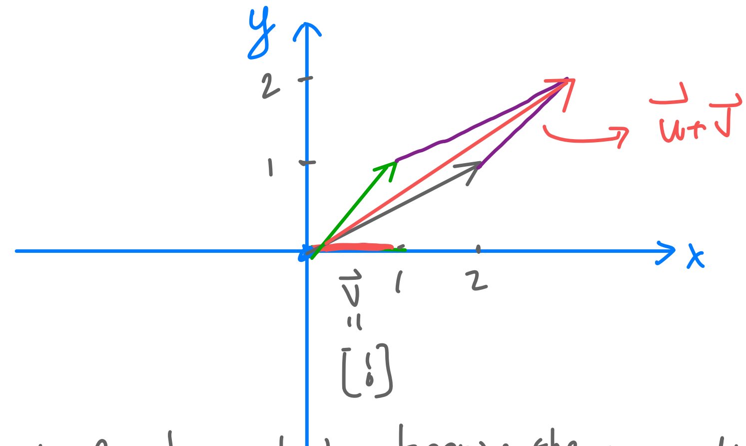
· geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Today: · the $\square$-law for vector addition · more geometry of solns of SELs.

· (Parallelogram law) For $\vec{u}, \vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$, $\vec{u}+\vec{v}$ corresponds to the (arrow from the origin to) the fourth vertex of the parallelogram ($\square$) where the vertices are 0, $\vec{u}$, and $\vec{v}$.

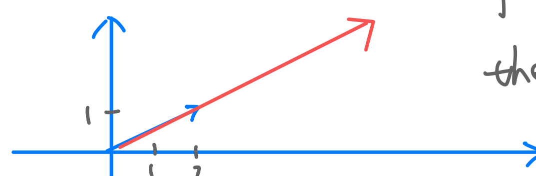
e.g.

$$\underset{\vec{u}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \underset{\vec{v}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \longleftrightarrow$$



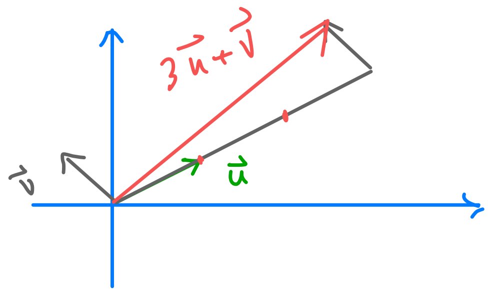
· Scaling a vector by $c \in \mathbb{R}$ scales the length by $|c|$, keeping the same direction if $c \geq 0$ and reversing the direction if $c < 0$.

e.g. $3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. $\longleftrightarrow$



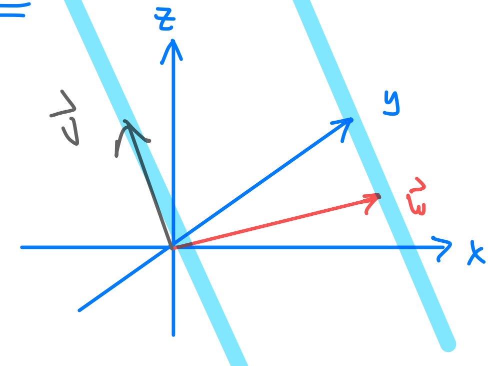
· (Linear combinations) Now that we can add and scale vectors, we can draw lin comb. of vectors as well.

e.g.



$3\vec{u}+\vec{v}$?

E.g. Given two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, the set $\{ z\vec{v}+\vec{w} \mid z \in \mathbb{R} \}$ can be described as the line in $\mathbb{R}^3$ that passes through $\vec{w}$ in the direction of $\vec{v}$.



More drawings in homework ...

2. Homogeneous vs. Non-homogeneous systems.

Def. An LES of the form
$$(*)\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$
is called <u>homogeneous</u> if $b_1 = b_2 = \cdots = b_m = 0$. Similarly, we say a vector equation $x_1\vec{v}_1 + \cdots + x_k\vec{v}_k \underset{(*)}{=} \vec{v}$ or a matrix equation $C\vec{x} \underset{(*)}{=} \vec{v}$ is <u>homogeneous</u> if $\vec{v} = 0$.

Note:

$\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a soln of $*)$ iff $(*)$ is homogeneous.

E.g.

$$\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

C (coefficient part), $\vec{b}$ (last column); columns of the R.E.F.: b b f

R.E.F.

(1) The eq. $C\vec{x}=\vec{b}$ is not homogeneous.

Recall its soln set is $X := \left\{ z\begin{bmatrix}1\\-2\\1\end{bmatrix} + \begin{bmatrix}-2\\3\\0\end{bmatrix} : z \in \mathbb{R} \right\}$.

Note: $\begin{bmatrix}-2\\3\\0\end{bmatrix}$ is a particular soln of $C\vec{x}=\vec{b}$.

Geometrically, X is a line in $\mathbb{R}^3$ going through $\begin{bmatrix}-2\\3\\0\end{bmatrix}$ in the direction of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

(2). The eq. $C\vec{x}=\vec{0}$ is homogeneous. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 0 \\ 9 & 10 & 11 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Its soln set is $X' := \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : \begin{array}{l} x-z=0 \\ y+2z=0 \\ z\in\mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix}z\\-2z\\z\end{bmatrix} : z\in\mathbb{R} \right\} = \left\{ z\begin{bmatrix}1\\-2\\1\end{bmatrix} : z\in\mathbb{R} \right\}$.

Geometrically, X' is a line in $\mathbb{R}^3$ going through the origin in the direction of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

Thm. Consider a hom. matrix eq. $C\vec{x} = \vec{0}$ and a non-hom matrix eq. $C\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$. Suppose both equations are consistent. Let $S_{\vec{0}}$ be the soln set of $C\vec{x} = \vec{0}$ and let $S_{\vec{b}}$ be the soln set of $C\vec{x} = \vec{b}$.

Let $\vec{v}$ be <u>any</u> particular soln of $C\vec{x} = \vec{b}$. ($\begin{bmatrix}-2\\3\\0\end{bmatrix}$ in the last e.g.)

Then we have $S_{\vec{b}} = S_{\vec{0}} + \vec{v}$, i.e.,

$$S_{\vec{b}} = \{ \vec{w} + \vec{v} \mid w \in S_{\vec{0}} \},$$

So geometrically $S_{\vec{b}}$ is a shift of $S_{\vec{0}}$.

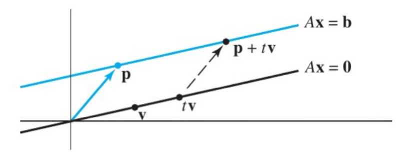


FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

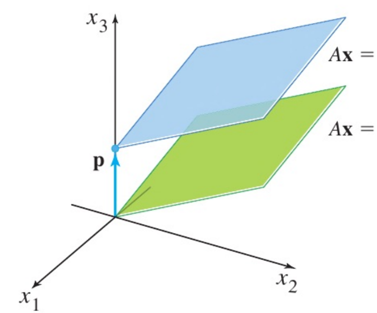


FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Ex. (Challenge). Prove this. **Key:** $C\vec{x} + C\vec{y} = C(\vec{x} + \vec{y})$ for matrix-vector products.

Ex. Work out the soln sets of $\begin{cases} x+2y-z=4 \\ x-y+z=0 \end{cases}$

and $\begin{cases} x+2y-z=0 \\ x-y+z=0. \end{cases}$ Draw them, compare them and note that they fit the descriptions of the theorem.

Hom.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix} \to S_{\vec{0}} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x+\frac{1}{3}z=0 \\ y-\frac{2}{3}z=0 \end{array} \right\} = \left\{ \begin{bmatrix} -\frac{1}{3}z \\ \frac{2}{3}z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$\begin{array}{ccc} b & b & f \end{array}$$

$$= \left\{ z \cdot \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}.$$

Nonhom. ($\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$)

Next time: lin. transformations

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 2 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \end{array}\right] \to \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x + \frac{1}{3} z = \frac{4}{3} \\ y - \frac{2}{3} z = \frac{4}{3} \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} \frac{4}{3} - \frac{1}{3} z \\ 4/3 + \frac{2}{3} z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 4/3 \\ 4/3 \\ 0 \end{bmatrix}}_{S_{\vec{b}}} : z \in \mathbb{R} \right\}.$$

So, $\begin{cases} S_{\vec{0}} \\ S_b \end{cases}$ is the line in $\mathbb{R}^3$ going through $\begin{cases} \text{the origin} \\ \begin{bmatrix} 4/3 \\ 4/3 \\ 0 \end{bmatrix} \end{cases}$ in the direction of $\begin{bmatrix} -1/3 \\ 2/3 \\ 0 \end{bmatrix}$.

In particular, they are shifts of each other. $\square$.