Math 2135. Lecture 8.

09. 10. 2021.

Last time: Let $S = \{\vec{v_1}, \dots, \vec{v_k}\} \subseteq \mathbb{R}^n$ and let $C = \left[\vec{v_1} \middle| \cdots \middle| \vec{v_k}\right]$.

(1). proved "S spans $\mathbb{R}^n \iff EF(C)$ has no zero rows".

(2). defined lin. ind.: S is lin ind. if $C\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$.

(3). proved echelon form criterion for lin ind: S is lin. ind $\iff$ every col in C is pivot.

(4) claimed: $\begin{cases} \text{if } k > n, \text{ then } S \text{ cannot be lin ind.} \\ \text{if } k < n, \text{ then } S \text{ cannot span } \mathbb{R}^n. \end{cases}$

Today:

- more criteria for lin ind.

- geometry of vectors

1. Non-EF criteria for linear ind.

Let $S = \{\vec{v_1}, ---, \vec{v_k}\} \in \mathbb{R}^n$.

Prop 2. (a). If $\vec{0} \in S$, then S is not lin ind.

Example: $S = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\} \subseteq \mathbb{R}^3$. $\underbrace{2 \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\2\\3\end{bmatrix}} = \vec{0}$ nontrivial ln comb.

pf: Suppose $S = \{\vec{v_1}, \dots, \vec{v_k}\}$ contains $\vec{0}$.

a similar trick works if some other v_i is zero

Without loss of generality (WLOG), we may assume that $\vec{v_1} = \vec{0}$.

Then $1 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + \dots + 0 \cdot \vec{v_k} = \vec{0}$ is a nontrivial ln comb of S that equals $\vec{0}$. so S is not lin ind. □

(b) If $|S| = 1$, i.e., $S = \{\vec{v_1}\}$ for a vector $\vec{v_1}$, then S is lin ind. iff $\vec{v_1} \neq \vec{0}$.

pf: (1) if $\vec{v} \neq 0$, then $\vec{v} = \begin{bmatrix}b_1\\ \vdots \\ b_n\end{bmatrix}$ where $b_i \neq 0$ for some $1 \leq i \leq n$. ... , so S is lin ind.

see next page

(2) if $\vec{v} = 0$, then S is not lin. ind. by (a). □

The ... part :

$\vec{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \neq \vec{0}$. So some $b_i \neq 0$.

To prove lin ind of S. take $c \in \mathbb{R}$ s.t.

$$c \cdot \vec{v} = 0$$

$$\Rightarrow c \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow cb_i = 0 \xRightarrow{b_i \neq 0} c = 0 .$$

So $c = 0$, therefore $S = \{\vec{v}\}$ is lin ind.

(c). If $|S| = 2$, say $S = \{\vec{v_1}, \vec{v_2}\}$, then

S is lin ind $\Longleftrightarrow$ neither elt in S is a multiple of the other.

pf: ($\Leftarrow$) Suppose neither elt in S is a multiple of the other. $(*)$

We need to prove that S is lin ind.

Assume $c_1\vec{v_1} + c_2\vec{v_2} = 0$ for some $c_1, c_2 \in \mathbb{R}$.

We claim that $c_1 = c_2 = 0$. Otherwise, say, WLOG, $c_1 \neq 0$.

so $c_1\vec{v_1} = -c_2\vec{v_2}$ and $\vec{v_1} = -\frac{c_2}{c_1} \cdot \vec{v_2}$, contradicting $*$.

It follows that S is lin ind.

($\Rightarrow$). Suppose S is lin ind.

($\triangle$)

We claim that ($*$) holds. Otherwise, WLOG we may assume $\vec{v}_1 = c\vec{v}_2$ for some $c \in \mathbb{R}$.

If $c = 0$, then $\vec{v}_1 = 0$, so S contains 0, so S is lin dep; contradicting ($\triangle$).

If $c \neq 0$, then $\vec{v}_1 - c\vec{v}_2 = 0$ is a nontrivi lin comb of S that equals 0, contradicting ($\triangle$).

It follow that ($*$) holds. $\square$.

Thm 3. A set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is linearly dependent iff some elt in S is a lin. comb. of the others.

Let's see some example applications first.

(a) $$S_1 = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} \right\}.$$

$\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ by inspection,

So S_1 is not lin ind.

(b) $$\begin{array}{ccc} \left[\begin{matrix} 4 \\ -7 \\ -3 \end{matrix}\right. & \begin{matrix} 1 \\ 5 \\ 3 \end{matrix} & \left.\begin{matrix} 0 \\ 3 \\ 3 \end{matrix}\right] \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\rightsquigarrow \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 = 0 \implies \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. dep.

eg. $\vec{v}_3 = \vec{v}_1 + 2v_2$

Note: If Prop 2 and Thm 3 are applicable, we can use them to detect lin dep/ind.
If not, we can use the EF Criterion.

pf of Thm 3: Ex.

The claim from the last lecture: Let $S = \{\underbrace{\vec{v_1}, \dots, \vec{v_k}}_{\text{all distinct}}\} \subseteq \mathbb{R}^n$, $C = [\vec{v_1} | \cdots | \vec{v_k}]$.

(1) $k > n \Rightarrow S$ is lin dep; equivalently, S is lin ind $\Rightarrow k \leq n$

(2) $k < n \Longrightarrow S$ doesn't span $\mathbb{R}^n$; equivalently, S spans $\mathbb{R}^n \Rightarrow k \geq n$.

Pf: Idea: Consider E.F.

(1) S is lin ind $\Rightarrow$ every col in C is pivot

$$\Rightarrow n = \boxed{\# \text{rows} \geq \# \text{pivots}} = \# \text{cols} = k$$

Since def. of pivot implies a row has at most one pivot

(2) S spans $\mathbb{R}^n \Rightarrow$ every row in C is pivot

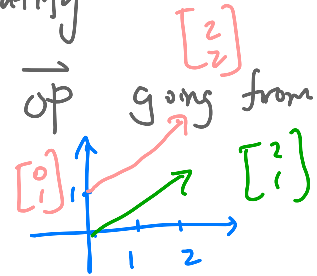
$$\Rightarrow n = \# \text{rows} = \boxed{\# \text{pivots} \leq \# \text{cols}} = k.$$

since def of EF implies a col has at most one pivot.

2. Geometry of Vectors.

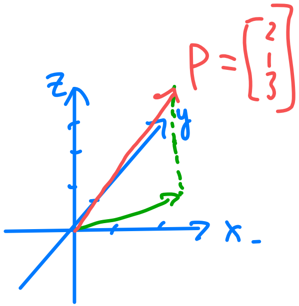
Next time: ▱-law for vec. addition; geometry of solns for SELs.

- We identify $\mathbb{R}^2$ with the 2-D plane and identify each vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ with the arrow $\overrightarrow{OP}$ going from the origin O to the point $P = (a.b)$ e.g. $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftrightarrow$



- We identify $\mathbb{R}^3$ with the 3D space and identify each vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ with the arrow $\overrightarrow{OP}$ going from the origin O to the point $P = (a. b, c)$

e.g. $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \leftrightarrow$



- We also identify arrows with the same direction and length as equal vectors.