Math 2135. Lecture 7.

09.08.2021.

Last time. Spanning properties via Echelon forms:

Let $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ and let $C = \left[\, \vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_k \,\right]$. Then

(1). We have $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for a given vec. $\vec{v} \in \mathbb{R}^n$ if and only if the matrix $A = [\, C \mid \vec{v} \,]$ doesn't have a pivot in the last column, i.e., $EF(A)$ has no row of the form $[0\ 0 \cdots 0 \mid b]$, $b \neq 0$.

(2) We have $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for all vectors $\vec{v} \in \mathbb{R}^n$ if and only if $EF(C)$ has no zero row.

Today: Pf of (2); recall that we've proved (1); linear independence.

1. Pf of (2) $\left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n, \quad C = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_k] \right)$

Let $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^n$.

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Consider the equation $C \cdot \vec{x} = \vec{v}$

the augmented matrix $A = [C | \vec{v}] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 2 & 3 & 5 & b_2 \\ 3 & 5 & 8 & b_3 \end{array}\right]$

We row-reduce A: $A \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 2 & 2 & b_3 - 3b_1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & ? \end{array}\right]$

Note: The entries in the last column are always lin. comb. of $b_1, b_2, \ldots, b_n$.

? $(b_3 - 3b_1) - 2(b_2 - 2b_1) = b_1 - 2b_2 + b_3$

Pf: We note that for each $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$, we have

$$EF\left(\left[C \,\middle|\, \vec{v} \right] \right) = \left[EF(C) \,\middle|\, \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{matrix} \right]$$ where $c_1, c_2, \cdots, c_n$ are

linear expressions in $b_1, b_2, \cdots, b_n$.

<u>Now</u>, (a) if $EF(C)$ has a zero row, then the last row (nth row) must be a zero row. For some $b_1, \cdots, b_n$, we must have $c_n \neq 0$. In this case, the last row of $EF([C|\vec{v}])$ is of the form $[0 \cdots 0 | \underset{\neq 0}{c_n}]$, hence for the corresponding $\vec{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, we have $\vec{v} \notin \mathrm{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$.

(b) Conversely, if $C\vec{x} = \vec{v}$ is incosistent for some v, then $EF([C|\vec{v}])$ has a row of the form $[0 \cdots 0 | b]$, $b \neq 0$, so $EF(C)$ has a zero row.

Combining (a) and (b), we see that

$$\vec{v} \notin \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} \text{ for some } \vec{v} \in \mathbb{R}^n \iff EF(C) \text{ has a zero row.}$$

It follows that

$$\vec{v} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} \text{ for all } \vec{v} \in \mathbb{R}^n \iff EF(C) \text{ has no zero row.}$$

$\square$

2. Linear independence

This doesn't have to be the case: e.g. $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\4\end{bmatrix}$, $c_1 = 2$, $c_2 = -1$ then $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ without $c_1 = c_2 = 0$.

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \in \mathbb{R}^n$.

Def We say S is <u>linearly independent</u> or <u>$\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent</u> if in order to have $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$ for $c_1, c_2, \ldots, c_k \in \mathbb{R}^n$ we have must have $c_1 = c_2 = \cdots = c_k = 0$. If S is not lin. ind, we say it is <u>linearly dependent</u>.

Def. (trivial lin. comb.) The linear comb. $0 \cdot \vec{v}_1 + \cdots + 0\vec{v}_k$ is called the trivial linear comb. of $v_1, \ldots, v_k$.

So, a set of elts in $\mathbb{R}^n$ is lin. ind. iff the only linear comb of them that equals zero is the trivial one.

$\to$ We can rephrase this via vec/matrix equations

Thm 1. Let $S = \{\vec{v_1}, \cdots, \vec{v_k}\} \subseteq \mathbb{R}^n$ and $C = [\vec{v_1} \mid \vec{v_2} \mid \cdots \mid \vec{v_k}]$. Then

TFAE:

(1) S is lin. ind.

$\Updownarrow$ ✓

(2) The vec eq. $x_1\vec{v_1} + \cdots + x_k\vec{v_k} = \vec{0}$ has a unique soln, namely, the trivial/obvious one w/ $x_1 = \cdots = x_k = 0$.

$\Updownarrow$ ✓

(3) The mat. eq. $C\vec{x} = \vec{0}$ has a unique soln.

$\Updownarrow$ ✓ "uniqueness of solns"

(4). $EF(C)$ has a pivot in every column. i.e., "$EF(C)$ has no free variables".

Note: The equivalence (1) $\Leftrightarrow$ (4) gives an echelon form criterion for lin. independence.

Example.

(1) $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$. Is S a spanning set of $\mathbb{R}^2$?

Is S lin independent?

Soln.

$$EF\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right) = \begin{array}{c} \begin{array}{cc} b & b \end{array} \\ \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \end{array}$$

Since the echelon form has no row of zeros, S spans $\mathbb{R}^2$.

Since all columns in the EF are pivot, S is lin. ind.

(2). $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$. Does it span $\mathbb{R}^2$? Is it lin ind?

Informal sketch:

Method 1 (EF):

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \to \begin{array}{cc} b & f \\ \end{array} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

zero row $\Rightarrow S$ doesn't span $\mathbb{R}^2$

a nonpivot column $\Rightarrow S$ is not lin. ind.

Method 2 (def.)

a line

$$\operatorname{Span} S = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}_{x, y \in \mathbb{R}} = \left\{ z \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} : z \in \mathbb{R} \right\} \neq \mathbb{R}^2,$$

so S doesn't span $\mathbb{R}^2$.

$$3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow \underbrace{3}_{\neq 0} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 0$$

S is lin. dep.

13). $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. Does S span $\mathbb{R}^3$? Is S lin ind?

Soln.

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \to \begin{array}{c} \begin{array}{cc} b & b \end{array} \\ \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

E.F.

The echelon form has a zero row, so S does not span $\mathbb{R}^3$.

All cols in the E.F are pivot, so S is lin ind.

Ex: (a). Check the lin ind. of S by solving $x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b). Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$.

- if $k < n$, then S does not span $\mathbb{R}^n$
- if $k > n$, then S is not lin ind.

Next time:

→ proofs. more criteria for lin ind.