Math 2135. Lecture 6.

09.03.2021.

Last time: · p.v.f. of LES solns.

· LES as vector and matrix equations.

Today. · spans and related questions (we'll try to answer them via echelon forms as usual.)

Def: (the span of a set of vectors in $\mathbb{R}^n$) Let $S=\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$ for some $n\geq 1$. The span of S or the span of $\vec{v}_1, \cdots, \vec{v}_k$ is the set of all possible linear combinations of $\vec{v}_1, \cdots, \vec{v}_k$, i.e., it's the set

$$\mathrm{Span}(\{\vec{v}_1, \cdots, \vec{v}_k\}) := \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, c_2, \cdots, c_k \in \mathbb{R}\}.$$

Some examples.

(1). Consider the set $\mathbb{R}$ of real numbers and take $\vec{v} \in \mathbb{R}$. (So $\vec{v}$ is just a number $[a_1]$)

if $\vec{v} = 0$ $([0])$, then $\mathrm{Span}(\vec{v}) = \{c \cdot [0] \mid c \in \mathbb{R}\} = \{[0]\}$ — it's a point, the origin

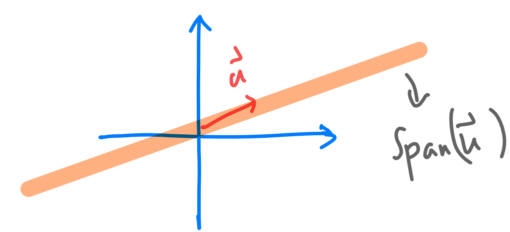
if $\vec{v} \neq 0$, (say $\vec{v} = [d]$, $d \neq 0$), then $\mathrm{Span}(\vec{v}) = \{[cd] : c \in \mathbb{R}\} = \{[x] : x \in \mathbb{R}\}$

$= \mathbb{R}$ (the entire real line)

(2) Consider the plane $\mathbb{R}^2$, and take $\vec{u}, \vec{v} \in \mathbb{R}^2$.

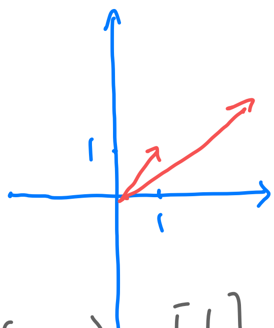
if $\vec{u} = \vec{0}$, then $\mathrm{Span}(\vec{u}) = \{\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$. (point)

if $\vec{u} \neq \vec{0}$, then $\mathrm{Span}(\vec{u}) = \{c\vec{u} \mid c \in \mathbb{R}\} \rightarrow$ the line in the direct (or the opp. direction of $\vec{u}$)



If $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

$$\operatorname{Span}(\{\vec{u}, \vec{v}\}) = \left\{ x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} : x, y \in \mathbb{R} \right\} \overset{\text{claim}}{=} \mathbb{R}^2$$



The claim, put another way:

the vec eq. $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a soln for all $a, b \in \mathbb{R}$.

If $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$$\operatorname{Span}(\{\vec{u}, \vec{v}\}) = \left\{ x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x + 2y \\ x + 2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} \rightarrow$$ line in the common dir of $\vec{u}, \vec{v}$.

Problem 1: Span membership.

Eg. Take $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

Note: The following questions are all equivalent:

- Is $\vec{v}$ in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$?

- (vec. eq.) Does the eq. $x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ have a soln?

- (matrix eq.) Does the eq. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ have a soln?

- (lin. sys.) $\begin{cases} x + 2y = 1 \\ 2x + 5y = 1 \\ y = 1 \end{cases}$

We know the answer!

(in terms of the aug. matrix $EF\left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$).

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\rightarrow$ no soln, so $\vec{v} \notin \mathrm{Span}(\{\vec{v}_1, \vec{v}_2\})$.

More generally ...

Thm 1. Let $A = [\, C \mid \vec{v} \,]$ where $C = [\, \vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_k \,]$.

Then the following are equivalent (T.F.A.E.):

(a) $\vec{v} \in \mathrm{Span}(\{\vec{v}_1, \ldots, \vec{v}_k\})$

(b) The vec equation $x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{v}$ has a soln.

(c) The matrix eq. $C\vec{x} = \vec{v}$ has a soln.

(d) The LES with aug. matrix A has a soln.

$\Updownarrow$ known

(e). Some echelon form of A has no row of the form $[0\ 0 \cdots 0\ b]$ where $b \neq 0$.

$\Updownarrow$

(e') The last col of A is not pivot.

Eg. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \in \mathbb{R}^3$ and $\vec{v} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \in \mathbb{R}^3$.

Q: $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$?

Soln: We row-reduce the matrix $A = [\, \vec{v}_1 \mid \vec{v}_2 \mid \vec{v} \,]$.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since EF(A) has no row of the form $[0 \; 0 \cdots 0 \; b]$ where $b \neq 0$, E.F.

it follows (from Thm 1) that $\vec{v} \in \operatorname{Span}(\{\vec{v}_1, \vec{v}_2\})$. □

Problem 2. Spanning sets of $\mathbb{R}^n$

Q: How can we tell if a given set spans $\mathbb{R}^n$?

Def (Spanning sets) A set of vectors $S \subseteq \mathbb{R}^n$ is said to span $\mathbb{R}^n$ / be a spanning set of $\mathbb{R}^n$ if $\mathrm{Span}(S) = \mathbb{R}^n$.

Examples / Nonexamples: By the examples on Page 2,

- $S = \{\vec{v}\} = \{[d]\} \subseteq \mathbb{R}^1$ spans $\mathbb{R}^1$ iff $d \neq 0$.
- The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$ by our claim; $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ does not span $\mathbb{R}^2$.
- Claim 2: The set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$. Reason: $\forall a, b \in \mathbb{R}$, $\begin{bmatrix} a \\ b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thm. Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ and let $C = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_k]$

Then TFAE. (say $v_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$)

(a) S is a spanning set of $\mathbb{R}^n$.

(b) The vec. eq. $x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{v}$ has a soln <u>for all $\vec{v} \in \mathbb{R}^n$</u>.

(c) The matrix eq. $C \cdot \vec{x} = \vec{v}$ has a soln <u>for all $\vec{v} \in \mathbb{R}^n$</u>.

(d) The lin system $\begin{cases} a_{11}x_1 + \cdots + a_{1k}x_k = b_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nk}x_k = b_k \end{cases}$ has a soln <u>$\forall\, b_1, b_2, \dots, b_k \in \mathbb{R}$</u>.

nontrivial, to be proved

(e) Some echelon form of A has no row of zeros.

(e') Every row of A $\exists$ pivot.

Pf: We've argued (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ ... $\Leftrightarrow$ (d). Need (d) $\Leftrightarrow$ (e). → next time.

Example. (1) Do $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ span $\mathbb{R}^3$?

i.e., Is $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a spanning set of $\mathbb{R}^3$?

Soln: $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{earlier}} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$EF([\vec{v}_1 | \vec{v}_2 | \vec{v}_3])$ has a zero row, so S is not a spanning set of $\mathbb{R}^3$.

(2). Ex: Does the set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ span $\mathbb{R}^2$?

Next time: pf. of Thm 2.

linear independence.