Math 2135. Lecture 5.

Hw1. due tonight 11:59 pm.

09.01.2021.

Last time: • Number of solns for a LES from an echelon form of the aug. matrix.

e.g. $\left[\begin{array}{cc|c} 0 & 3 & 7 \\ 2 & 6 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 6 & 10 \\ 0 & 3 & 7 \end{array}\right]$

consistent, unique soln

e.g. $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$

(similar to Hw problem)

if $6-3h=0$, i.e., if $h=2$, then the sys. has no soln; otherwise, there's a unique soln.

• To get the exact sols, use REF.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \longrightarrow \begin{cases} x = -2 \\ y = \frac{7}{3} \end{cases}$$

More generally, we'll write the solns in parametric vector form, as

$$\left\{ \begin{bmatrix} 3-y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}$$

• Linear comb. of vectors. e.g.

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$.

Today. New notation and terminology

1. Parametric vector form for solns of LES.

If an LES has inf. solns. we can write the solns as linear combinations of constant vectors where all but possibly one coeff is a free variable.

This is called the

e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} z-2 \\ 3-2z \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

"free variables appear as coefficients in the p.v.f."

Ex. Find the soln set of $\begin{cases} x - 3y - 5z = 0 \\ y - z = -1 \end{cases}$ in P.V.F.

Soln: The aug. matrix is $A = \left[\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right]$ $\xrightarrow{E3}$ $R1 \leftarrow R1 + 3R2$

Gaussian elimination shows that a red. EF of A is

$$\mathrm{REF}(A) = \begin{array}{cccc} x & y & z & \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} -8 \\ -1 \end{array} & \left|\begin{array}{c} -3 \\ -1 \end{array}\right] \\ b & b & f & \end{array}$$

So the soln set is $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \ \underline{z \in \mathbb{R}},\ x - 8z = -3,\ y - z = -1 \right\}$

$$= \left\{ \begin{bmatrix} -3+8z \\ -1+z \\ z \end{bmatrix} \middle| \ z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 8z \\ z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\} \leftarrow \text{p.v.f}$$

$\square$

More examples to come ...

2. Vector equations

Since a vector stores multiple numbers at a time, we can encode a lin. equation system by one vector eq, i.e., an eq. of the form $x_1\vec{v}_1 + x_2\cdot\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$ where $x_1, x_2, \dots, x_n$ are variables and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{b}$ are constant vectors.

e.g. $\begin{cases} 2x = 10 \\ x + 5y = 20 \end{cases} \longleftrightarrow \begin{bmatrix} 2x \\ x+5y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \longleftrightarrow x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

e.g. $\begin{cases} x - z = 3 \\ y + 2z = 5 \\ x + y - 3z = 7 \end{cases} \longleftrightarrow x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

"variables in the lin sys. $\updownarrow$ coefficients in the vec. eq."

Translation to/from augmented matrices.

e.g.

$$x_1\begin{bmatrix}3\\1\\4\end{bmatrix}+x_2\begin{bmatrix}-1\\2\\3\end{bmatrix}=\begin{bmatrix}10\\1\\0\end{bmatrix} \leftrightarrow \begin{cases}3x_1-x_2=10\\x_1+2x_2=1\\4x_1+3x_2=0\end{cases} \leftrightarrow \begin{array}{c}\begin{array}{ccc}x_1 & x_2 & \end{array}\\ \left[\begin{array}{cc|c}3 & -1 & 10\\1 & 2 & 1\\4 & 3 & 0\end{array}\right]\end{array}$$

vector equation

encoding matrix.

generally.

the coeff in the vec eq. (i.e., the variables) $\longleftrightarrow$ the labels of the cols of the aug. matrix

the constant vectors in the vec. eq. $\longleftrightarrow$ the cols of the aug. matrix.

3. Matrix - vector products. (will give yet another way to write linear systems)

Let $A = \left[\vec{c}_1 \middle| \vec{c}_2 \middle| \cdots \middle| \vec{c}_p\right]$ be a matrix and let $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}$ be a vector.

We can sometimes (not always) define a product $A \cdot v$:

Def: if $p = r$, i.e., if # cols in A = # rows in $\vec{v}$

then we define the matrix-vector product $A \cdot \vec{v}$ to be the vector

$$A \cdot v = a_1 \vec{c}_1 + a_2 \vec{c}_2 + \cdots + a_p \vec{c}_p = \sum_{i=1}^{p} a_i \vec{c}_i$$

e.g.: $\begin{bmatrix} 2 & 4 & 0 \\ 5 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ is not defined.

$$\begin{bmatrix} 2 & 4 & 0 \\ 5 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \overset{\text{def}}{=} 1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 0 \\ 1 \cdot 5 + 2 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

size 2

(reverse direction:)

$$3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\underbrace{x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 5 \end{bmatrix}}_{\text{typical form for the ``left'' side of vector equations}} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

another way of understanding matrix-vector product:

$$\begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix} \cdot \vec{v} = \begin{bmatrix} \vec{r}_1 \cdot \vec{v} \\ \vec{r}_2 \cdot \vec{v} \\ \vdots \\ \vec{r}_p \cdot \vec{v} \end{bmatrix}$$

where "$\cdot$" denotes the usual inner/dot product.

Note: if lin. comb. of vectors can be written as matrix-vector products, then we can write a vector eq. $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ as a so-called <u>matrix equation</u> $A\cdot\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \vec{b}$ where $A = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n]$.

e.g.
$$\begin{cases} x - 2y = 5 \\ 3x + y = 6 \end{cases} \longleftrightarrow x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \longleftrightarrow \underbrace{\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 3 & 1 & 6 \end{array}\right]}_{\text{aug. matrix}}.$$

e.g.
$$x\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - y\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 3 & -4 & 1 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\Updownarrow$$

$$\begin{cases} x - 2y = 0 \\ 3x - 4y + z = 1 \\ 5x - 6y + 2z = 0. \end{cases}$$

More generally, we can translate between the traditional, vec eq., and matrix eq. forms of a lin system easily as follows

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \longleftrightarrow \text{vec. eq.} \quad x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$\updownarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\swarrow\!\!\nearrow$

matrix eq.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

aug. matrix

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

E.g. Solve the matrix equation $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. $\left(\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$

Soln. The aug. matrix of the system is $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 2 & 4 & 8 & 3 \\ 4 & 8 & 16 & 5 \end{array}\right]$.

We can now find the soln for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ using our old methods. (E.X.)

Next time:
- Working w/ the new notations
- spans and spanning sets

Ex. Solve $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.