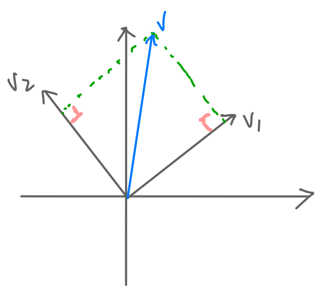
Math 2135. Lecture 41.

12.08. 2021.

Last time:

- orthogonality: $u \perp v \iff \langle u, v \rangle = 0$

- orthogonal sets / basis.
  - orthogonal sets are automatically lin. ind.
  - decomp. w.r.t. orthogonal bases are easy:



e.g. $C = \left\{ v_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ $(v_1 \perp v_2)$, $v = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$$\Downarrow$$

$$v = \underbrace{\frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2}_{} = \frac{22}{25} v_1 + \frac{21}{25} v_2$$

ortho. proj. of v onto v_1 and v_2.

Today:

- orthogonal projections
- obtaining orthogonal sets: the Gram-Schmidt process

1. Orthogonal projections.

Recall: If $\{v_1, \cdots, v_n\}$ is an ortho. basis of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then

$$v = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

, where each summand $\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} \cdot v_i$ is the orthogonal projection of v onto $\underbrace{\text{the line spanned by } v_i}_{\text{Span}(v_i)}$.

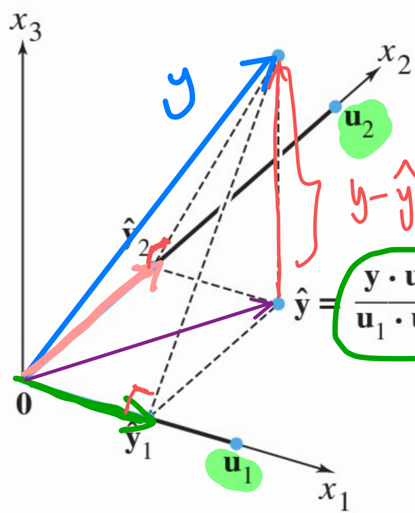
More generally, we can consider the orthogonal projection of an elt $v \in \mathbb{R}^n$ onto the span of any orthogonal set of vectors $\{v_1, \cdots, v_k\}$.

$y \in \mathbb{R}^n$

orthogonal set $\{v_1, \cdots, v_k\}$ $\Big\} \longrightarrow$

Def 1: Let $W = \text{Span}\{v_1, \cdots, v_k\}$. We define the orthogonal projection of v onto W as

$$\text{proj}_W\, y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$



$y - \hat{y} \perp \hat{y}$.

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$$

FIGURE 3 The orthogonal projection of **y** is the sum of its projections onto one-dimensional subspaces that are mutually orthogonal.

$$= \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2 = \text{proj}_{l_2} y$$

where $l_2 = \text{Span}(u_2)$

$$= \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 = \text{proj}_{l_1} y$$

where $l_1 = \text{Span}(u_1)$

$(\hat{y})$

$$= \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \text{proj}_W y$$

where $W = \text{Span}\{u_1, u_2\}$.

$(u_1 \perp u_2)$

Thm: Let y, $\{v_1, \dots, v_k\}$, and $W = \mathrm{Span}\{v_1, \dots, v_k\}$ be as in Def 1.

Let $\hat{y} = \mathrm{proj}_W y$. Then

(1) $y - \hat{y} \perp \hat{y}$ $\quad$ (so $y = \underbrace{(y-\hat{y})}_{\text{perp. to } W} + \underbrace{\hat{y}}_{W}$)

(2) $\hat{y}$ is the best approximation of / closest point to y in W in the sense that for all $w \in W$, we have $\|y - w\| \geq \|y - \hat{y}\|$.

Def: We call the length $\|y - \hat{y}\|$ in the above setting the distance between y and W.

Eg. Let $C = \Big\{ \underset{v_1}{\begin{bmatrix}1\\6\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-12\\2\end{bmatrix}} \Big\} \subseteq \mathbb{R}^2$.

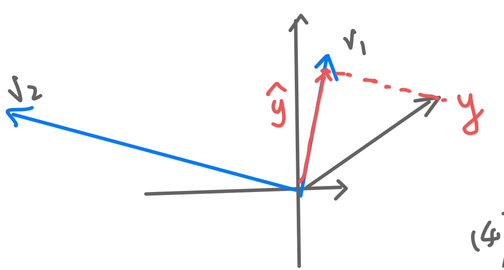
(1) Verify that C is an orthogonal basis of $\mathbb{R}^2$. EX.

(2) Find $[v]_C$ for $v = \begin{bmatrix}5\\5\end{bmatrix}$.

Soln: Since C is an ortho. basis,

$$v = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{35}{37} v_1 + \frac{-50}{148} v_2 = \frac{35}{37} v_1 - \frac{25}{74} v_2.$$

(3) Find the closest point to v on the line $\mathrm{Span}(v_1)$.



Soln: It's $\hat{y} = \mathrm{proj}_{\ell_1} y = \dfrac{y \cdot v_1}{v_1 \cdot v_1} v_1 = \boxed{\dfrac{35}{37} v_1} = \begin{bmatrix}35/37\\210/37\end{bmatrix}$.

where $\ell_1 = \mathrm{Span}(v_1)$

(4) Find the distance between y and the line spanned by v_2.

Ex. (should compute $\| \boxed{y - \mathrm{proj}_{\ell_2} y} \|$ where $\ell_2 = \mathrm{Span}(v_2)$.)

E.g. Let $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3$

(1) Prove that $u_1 \perp u_2$.

Ex.

(2) Let $W = \text{Span}\{u_1, u_2\}$. Find the distance from y to W.

Ex. The desired distance is $\|y - \hat{y}\| = \|y - \text{proj}_W y\| = \cdots\cdots$

2. The Gram-Schmidt process (won't be on the final)

We have seen many desirable properties of orthogonal bases.

The Gram-Schmidt process gives a way to produce orthogonal bases.

Input: A basis $\{x_1, \cdots, x_k\}$ of some subspace W of $\mathbb{R}^n$. (e.g. $\left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{x_1}, \underbrace{\begin{bmatrix}2\\1\\0\end{bmatrix}}_{x_2}, \underbrace{\begin{bmatrix}3\\4\\5\end{bmatrix}}_{x_3} \right\}$)

Output: An orthogonal basis $\{v_1, \cdots, v_k\}$ of W.

The process/algorithm (recursive):

(1) Initial step: set $v_1 = x_1$. ($v_1 = x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$.)

(2) When $v_1, \cdots, v_{i-1}$ have been found, set $U_i = \text{Span}\{v_1, \cdots, v_{i-1}\}$

and compute $\quad V_i = x_i - \text{proj}_{U_i} x_i$

(e.g. $x_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. $\longrightarrow U_2 = \text{Span}\, v_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

$\checkmark$

$V_2 = x_2 - \text{proj}_{U_2} x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$

$\checkmark$

(3) Repeat until all of $v_1, \cdots, v_k$ have been computed.

$x_1, x_2, x_3 \longrightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$

By repeated application of Thm. (1), the set $\{v_1, \cdots, v_k\}$ is orthogonal and hence an orthogonal basis of W. $\square$