Math 2135. Lecture 40. 12. 06. 2021.

Last time:

- properties of inner products (i.p.)
- geometry of i.p.
  - length: $\|v\| = \sqrt{v \cdot v}$
  - distance: $\text{dist}(u, v) = \|v - u\| = \|u - v\|$.
  - normalization: $v \longmapsto \underline{\frac{1}{\|v\|} \cdot v}$ $\rightarrow$ has length 1.

Today:

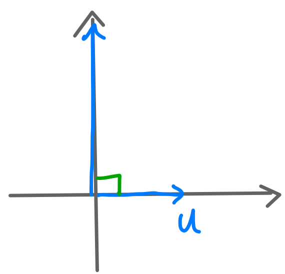
- orthogonality:
  - orthogonal sets and bases
  - orthogonal projections

1. Orthogonal sets

Let n be a positive integer.

Def: We say two vectors $u, v \in \mathbb{R}^n$ are <u>orthogonal</u> (to each other) if $u \cdot v = 0$. In this case, we write $u \perp v$.

E.g. (1)

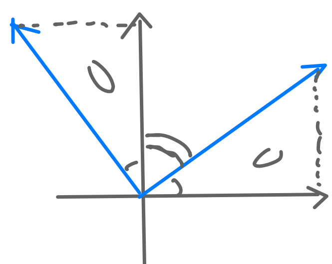


$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

→ geometry: $u \perp v$

→ algebra: $u \cdot v = 1 \cdot 0 + 0 \cdot 2 = 0$

(2)

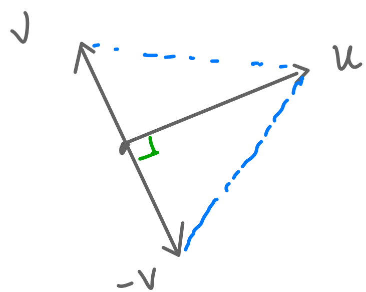


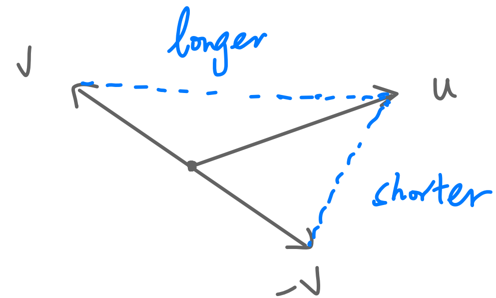
$$u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

→ geometry: $u \perp v$: $\alpha_{)} + \alpha_{))} = 90^\circ$

→ algebra: $u \cdot v = 4 \cdot (-3) + 3 \cdot 4 = 0.$

An explanation why the condition "$u \cdot v = 0$" is equivalent to "$u \perp v$":





Note: $u \perp v \iff \text{dist}(u,v) = \text{dist}(u,-v)$

$$\Updownarrow$$

$$\|u - v\| = \|u - (-v)\|$$

$$\Updownarrow$$

$$\|u - v\| = \|u + v\|$$

$$\Updownarrow$$

$$\|u-v\|^2 = \|u+v\|^2$$

$$\iff$$

$$\langle u-v, u-v\rangle = \langle u+v, u+v\rangle$$

$$\Updownarrow$$

$$-2\langle u, v\rangle = +2\langle u, v\rangle$$

$$\Updownarrow$$

$$\langle u, v\rangle = 0.$$

Def: (1) A set $C = \{v_1, v_2, \dots, v_k\}$ in $\mathbb{R}^n$ is called <u>an orthogonal set</u> if its elts are pairwise orthogonal, i.e., if $v_i \cdot v_j = 0 \ \forall \ 1 \le i < j \le k$.

(2) A basis of $\mathbb{R}^n$ is called an <u>orthogonal basis</u> of $\mathbb{R}^n$ if it is an orthogonal set.

E.g. (a) The standard basis $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \dots \right\}$ is a orthogonal basis of $\mathbb{R}^n$.

(b) The set $C = \left\{ \underset{v_1}{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}} \right\}$ is an orthogonal set in $\mathbb{R}^3$:

$v_1 \cdot v_2 = 3 \cdot (-1) + 1 \cdot 2 + 1 \cdot 1 = 0$, $\quad v_1 \cdot v_3 = 3 \cdot (-1/2) + 1 \cdot (-2) + 1 \cdot 7/2 = 0$,

$v_2 \cdot v_3 = (-1) \cdot (-1/2) + 2 \cdot (-2) + 1 \cdot 7/2 = 0$.

Why are orthogonal sets/bases interesting?

(1) An orthogonal set of nonzero elts is automatically linearly independent:

Prop 1: If a set $C = \{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is orthogonal, then C is lin. ind.

Note: Previously, to check if a set is lin. ind, we typically need to compute EFs.

E.g. Since the set $C \subseteq \mathbb{R}^3$ is orthogonal, it is linearly ind. by Prop 1.

Since C has exactly 3 elts and is lin ind, it follows that

C is a basis, hence an orthogonal basis, of $\mathbb{R}^3$.

(2) Computing coordinate vectors w.r.t orthogonal bases is easy:

Thm 2: Suppose $B = \{v_1, \ldots, v_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

Let $v \in \mathbb{R}^n$.

(**Note:** Recall that $v = c_1 v_1 + \cdots + c_n v_n$ for unique scalars $c_1, c_2, \ldots, c_n$, giving the coord. vector $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Also recall that to find $[v]_B$ we typically have to solve $x_1 v_1 + \cdots + x_n v_n = v$.)

Then $$v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots + \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} \cdot v_n.$$ easier

In other words, $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $c_i = \dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$ $\forall\ 1 \le i \le n$.

Eg: Consider the set $C = \left\{ \overset{v_1}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -3 \\ 4 \end{bmatrix}} \right\} \subseteq \mathbb{R}^2$ and the elt $v = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \in \mathbb{R}^2$.

(1) Check that C is a orthogonal basis of $\mathbb{R}^2$.

(2) Find $[v]_C$.

Soln: (1). $\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 4 \cdot (-3) + 3 \cdot 4 = 0$, so C is orthogonal.

By Prop 1, this implies C is lin. ind.

Since $|C| = 2$, it follows that C is a basis of $\mathbb{R}^2$.

(2) (old method: $x \begin{bmatrix} 4 \\ 3 \end{bmatrix} + y \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \rightarrow \left[\begin{array}{cc:c} 4 & -3 & 1 \\ 3 & 4 & 6 \end{array}\right] \rightarrow \cdots$)

By Thm 1,
$$[v]_C = \begin{bmatrix} \dfrac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} \\[2ex] \dfrac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} \end{bmatrix} = \begin{bmatrix} 22/25 \\[2ex] 21/25 \end{bmatrix}$$

Check: $22/25 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} + 21/25 \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 1/25 \cdot \begin{bmatrix} 88 - 63 \\ 66 + 84 \end{bmatrix} = 1/25 \cdot \begin{bmatrix} 25 \\ 150 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. ✓

Proofs. Suppose $C = \{v_1, \dots, v_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ and suppose $v \overset{*}{=} x_1 v_1 + \cdots + x_k v_k \in \mathbb{R}^n$. Then for each $1 \le i \le k$, "dotting with v_i" yields

$$\langle v, v_i \rangle = \langle x_1 v_1 + x_2 v_2 + \cdots + x_k v_k, v_i \rangle$$
$$= x_1 \langle v_1, v_i \rangle + \cdots + x_k \langle v_k, v_i \rangle$$
$$= x_i \langle v_i, v_i \rangle,$$

so $x_i = \dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$. $\xrightarrow{\text{Prop 1.}}$ Suppose $x_1 v_1 + \cdots + x_k v_k = 0$. Then $x_i = \dfrac{\langle 0, v_i \rangle}{\langle v_i, v_i \rangle} = 0$ for all i, so C is lin ind and Prop 1 holds.

$\downarrow$ Thm1

specialize: if $k = n$ and C is a basis, we get $[v]_C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ w/ $c_i = \dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$, so Thm 1 holds.

(3) One more reason: In the decomposition $v = c_1 v_1 + \cdots + c_n v_n = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i$ from the setting of Thm 1, the terms $\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i$ have geometric significance: they are "orthogonal projections".

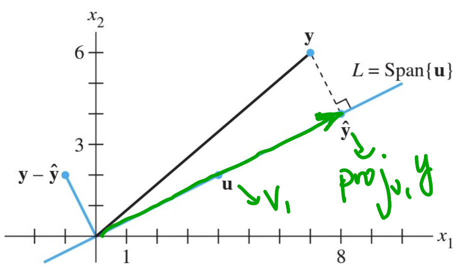


FIGURE 3 The orthogonal projection of **y** onto a line L through the origin.

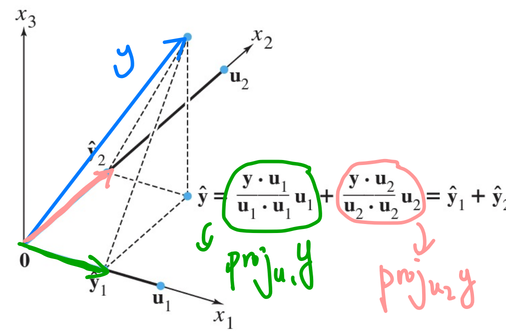


FIGURE 3 The orthogonal projection of **y** is the sum of its projections onto one-dimensional subspaces that are mutually orthogonal.

Next time:
- more on orthogonal projections.
- the Gram-Schmidt process.