Math 2135. Lecture 4.

08.30.2021.

So far: The following picture is almost completed

$$\begin{array}{ccc}
\text{given L.E.S} & \xrightarrow{\text{encode}} & \text{the augmented matrix } A = \left[\begin{array}{c|c} C & \vec{b} \end{array}\right] \\
\Big\downarrow & & \Big\downarrow \text{Gaussian Elimination w/ E1, E2, E3} \\
\text{the solns} & \xleftarrow[\text{decode}]{} & \text{EF/REF of } A
\end{array}$$

C: coefficient matrix

$\vec{b}$: last column

Today:

- How many solns are there to the L.E.S ?
- What are the solns

Answer these in terms of the (R)EF.

1. Number of solns from echelon form (Is the system consistent? If so, how many solns?)

Definitions and observations.

1. Def. (Pivot columns, pivot positions, basic variables, free variables)

Let A be a matrix and B be any echelon form of A.

(e.g. $A = \begin{array}{c} \begin{array}{cccc} b & b & f & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \end{array} \longrightarrow B = \begin{array}{c} \begin{array}{cccc} P & P & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$)

O : pivot positions

A pivot position in A is a position where there's a row-leading entry in B.

Such positions are independent of the choice of B.

A pivot row or column in A is a row or column in A with a pivot position.

For a L.E.S with augmented matrix $A = [\, C \mid \vec{b}\,]$, a variable is called a basic variable if its column is pivot and a free variable if its column is not pivot

Examples & observations

(1) $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{array}{cccc} x & y & z & \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 2 \\ 0 \end{matrix} & \left|\begin{matrix} -2 \\ 3 \\ 0 \end{matrix}\right] \\ b & b & f & \end{array}$

$$\text{Soln set} = \left\{ \underset{(x,y,z)}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} : \begin{matrix} x-z=-2 \\ y+2z=3 \end{matrix}, z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} z-2 \\ 3-2z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

Note: (1) The above system has inf. many solns since it has a free variable z, which ~~can~~ is "free" to take any value (to yield) a soln.

(2). $\begin{cases} x+y=3 \\ x+2y=7 \end{cases} \to \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 7 \end{array}\right] \to \begin{array}{c} \begin{array}{cc} b & b \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 4 \end{array}\right] \end{array} \to \begin{array}{c} \begin{array}{cc} b & b \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right] \end{array}$

EF! REF.

$\to \begin{cases} x=-1 \\ y=4 \end{cases}$ no free variable, unique soln.

(3). $\begin{cases} x+y=3 \\ 2x+2y=7 \end{cases} \to \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$

$\to$ nonsense: "$0x+0y=1$", can't happen. $\to$ the system is not consistent

(4) $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases} \to \left[\begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right] \to \begin{array}{c} \begin{array}{cc} b & f \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array} \to \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{c} x+y=3 \\ y \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} 3-y \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$

the sys. is consistent, and there are inf. many solns since there's a free var.

Roughly: "nonsense rows" $\leftrightarrow$ inconsistency

free variables $\leftrightarrow$ non-uniqueness

(5) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \end{bmatrix} \to \begin{array}{ccc} b & b & \\ \left[\begin{matrix} 1 & 3 \\ 0 & -4 \end{matrix}\right. & \left.\begin{matrix} 7 \\ -8 \end{matrix}\right] \end{array}$ $\left(\dashrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{cases} x = 1 \\ y = 2, \end{cases} \text{unique soln} \right)$

$\shortparallel$ B, in E.f.

we don't have the do this : B tells us that there should be a soln and a unique soln.

(6) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \end{bmatrix} \to \begin{array}{ccc} b & f & \\ \left[\begin{matrix} 1 & 3 \\ 0 & 0 \end{matrix}\right. & \left.\begin{matrix} 7 \\ 0 \end{matrix}\right] \end{array} \to$ inf. many solns, "y" is free.

EF

(7) $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \to \begin{array}{ccc} b & b & \\ \left[\begin{matrix} 2 & 6 \\ 0 & 3 \end{matrix}\right. & \left.\begin{matrix} 10 \\ 7 \end{matrix}\right] \end{array} \to$ unique soln.

EF

(8) $\begin{bmatrix} 2 & 6 & 3 \\ 1 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 \\ 2 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & -15 \end{bmatrix} \to$ nonsense row, the system is inconsistent.

assume the matrices are aug. matrices of lin. systems.

Prop 1. (Consistency / existence of solns)

Let A be the aug. matrix of a L.E.S. , and let B be any echelon form of A. Then the L.E.S is consistent if and only if B has no row of the form $[0 \ 0 \dots 0 \mid b]$ where $b \neq 0$.

Note: To tell if an L.E.S. is consistent, it suffices to get just any echelon form of its augmented matrix and not necessarily the R.E.F.

E.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \rightarrow$ consistent

Prop 2. (uniqueness / number of solns)

Say a L.E.S. has aug. matrix A. and assume that the L.E.S is consistent. Then the L.E.S has a unique soln iff all variables are basic, i.e., iff all variable columns in A are pivot.

Note: As before, here we don't need a R.E.F to tell the number of solns.

E.g.

$$A \to \begin{bmatrix} \boxtimes & * & * & * \\ 0 & \boxtimes & * & * \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

↓ inconsistent

$\boxtimes$ stands for a nonzero number

$$A \to \begin{array}{c} \begin{bmatrix} \boxtimes & * & * & * & * \\ 0 & 0 & \boxtimes & * & * \\ 0 & 0 & 0 & \boxtimes & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & f & & & \end{array} \end{array}$$

→ consistent.
inf. many solns.

E.g: Transform the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & -1 & 3 & 1 \\ 4 & -3 & 0 & 3 \end{bmatrix}$ into Echelon form, determine if the L.E.S. encoded by A is consistent, then find the soln set of the L.E.S.

$\exists$ unique soln

$\Uparrow$

Soln: $A \rightarrow \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 4 & -3 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -3 & 2 \end{bmatrix} \rightarrow \begin{array}{cccc} b & b & b; & \\ \end{array} B = \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

✓ E.F.

B is an E.F of A and has no row of the form $[0\ 0 \cdots 0\ b]$ with $b \neq 0$, so the LES is consistent by Prop 1.

The system has a unique soln by Prop 2 since all its cols are pivot.

To get the unique soln, we will use the REF.

Ex: Find the unique soln.

2. Parametric Vector Forms.

← We will write soln sets of lin. systems in this form.

Vectors.

just like numbers but multiple numbers at a time.

- Def: A <u>Vector</u> is a list of numbers arranged in a column. eg. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

- <u>Operations on Vectors</u>:

(addition). We can add two vectors of the same size, entry-wise.

eg $$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 2-1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}.$$

(scalar mult.) We can scale a vector by a number:

$$c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \qquad \text{eg. } 5 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}.$$

Def. Given vectors $\vec{v}_1 \ldots \vec{v}_k$ of the same size and numbers $c_1, c_2 \ldots c_k$, a vector of the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

is called a linear combination of $\vec{v}_1, \ldots \vec{v}_k$.

e.g. $3 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 5 - 2\cdot 1 \\ 3\cdot 1 - 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$.

So $\begin{bmatrix} 13 \\ 3 \end{bmatrix}$ is a lin. comb of $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

e.g. $\begin{bmatrix} 3-y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ("parametric vector form")

Next time:
more on p.v.f.
vectors, and matrices