Math 2135. Lecture 39.

Final Exam: Tues., Dec 14.
4:30 – 7:00 pm.

12.03.2021.

Last time:

- another interpretation of diagonalization

  "$A = PDP^{-1}$" $\rightarrow$ $[T_A]_B = D$ for $B = \{\text{cols. of } P\}$ → finishes Ch. 5.
  (mult. by A)

- inner product formula: $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

- properties of inner products.

§6.1.

Today:

- Pf of properties of inner products. (i.p.)

- geometric notions associated to inner products

  (a) length (b) distance (c) normalization (d) orthogonality

1. Properties of inner products

Prop. (stated last time) Let $u, v, w \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then

(1) $u \cdot v = v \cdot u$. commutativity

(2) $(u+v) \cdot w = u \cdot w + v \cdot w$ distributivity over $+$.

(3) $c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$

Note: (1), (2), (3) imply that the i.p. is "bilinear".

(4) $u \cdot u \geq 0$, with $u \cdot u = 0$ iff $u = \vec{0}$.

Pf: Suppose $u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $w = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then

(1) $u \cdot v = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} b_i a_i = v \cdot u$, so (1) holds.

(2) $(u+v) \cdot w = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} (a_i+b_i) \cdot c_i = \sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n} b_i c_i = u \cdot w + v \cdot w$.

13) E.X. (similar)

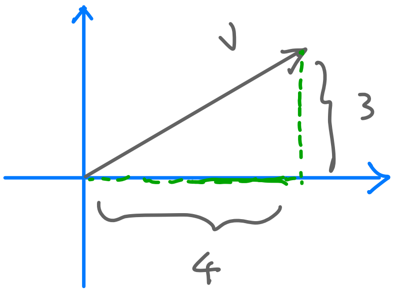
14). $u \cdot u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$ since $a_i^2 \geq 0 \ \forall i$.

Also, $u \cdot u = 0 \iff a_1^2 + \cdots + a_n^2 = 0 \iff a_i^2 = 0 \ \forall i \iff a_i = 0 \ \forall i \iff u = 0$. $\square$

2. Geometry of inner products

(a) Length

Eg.



$n=2$, $v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

By the Pythargorean Thm, we'd say the length of v is $\sqrt{\underline{4^2+3^2}} = 5$.

Note: this is $\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = v \cdot v$

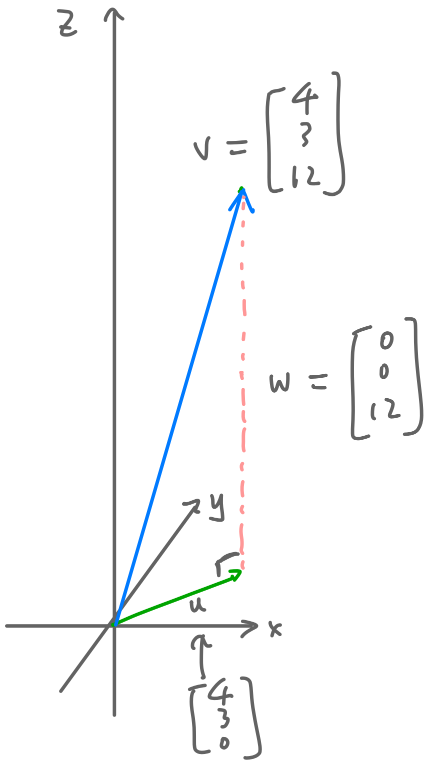
generalize

Def. We define the length of a vector $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{R}^n$ to be

$$\text{len}(v) = \|v\| := \sqrt{v \cdot v} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Rmk: Our example shows that the above def. of length agree with our intuition from Euclidean geometry.

E.g. $n = 3$, $v = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$.



By our def, $\|v\| = \sqrt{4^2 + 3^2 + 12^2} = \sqrt{169} = 13$.

Also, by geometry $\|v\| = \sqrt{\|u\|^2 + \|w\|^2}$

$$= \sqrt{(4^2 + 3^2) + 12^2}$$

$$= \sqrt{169} = 13.$$

So again, the def of $\|v\|$ captures Euclidean geometry.

(b) Distance

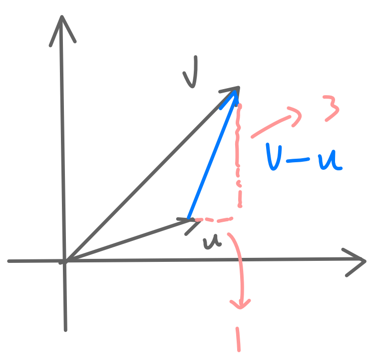
Def. Let $u, v \in \mathbb{R}^n$. We define the distance from u to v to be

$$\operatorname{dist}(u,v) := \|v - u\|.$$

E.g. $n=2$. $u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. $\Rightarrow$ $\operatorname{dist}(u,v) = \|v-u\|$

$$= \left\| \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2+3^2} = \sqrt{10}.$$



Note: For any vector $w \in \mathbb{R}^n$, we have $\|w\| = \|-w\|$, so

$$\operatorname{dist}(u,v) = \|v-u\| = \|u-v\| = \operatorname{dist}(v,u).$$

that is, $\operatorname{dist}(u,v) = \operatorname{dist}(v,u)$, both the "distance between u and v".

(c). Normalization

Def. A unit vector in $\mathbb{R}^n$ is a vector $u \in \mathbb{R}^n$ with $\|u\| = 1$.

Note: Given any nonzero vector $0 \neq v \in \mathbb{R}^n$, there is a unique vector v' that has the same direction as v and has length 1, i.e., there $\exists$ a unique unit vector in the direction of v.

$\Updownarrow$ (v' is a positive multiple of v)

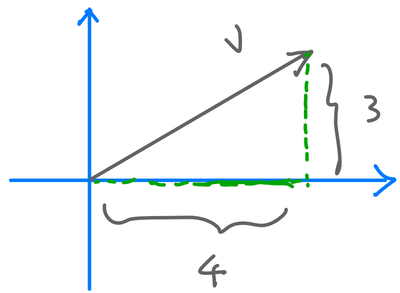
Def: In the above setting, we call v' the normalization of v.

Prop: (i) $\forall v \in \mathbb{R}^n$, $c \in \mathbb{R}_{>0}$, we have $\|cv\| = c\|v\|$.

(ii) For every $0 \neq v \in \mathbb{R}^n$, the normalization of v is given by

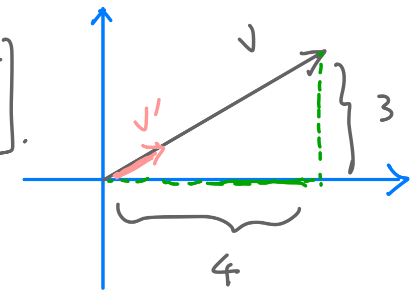
$$v' = \frac{1}{\|v\|} \cdot v.$$

Example for (ii).



, we saw $\|v\|=5$

$\Rightarrow v' = \frac{1}{5}\cdot v = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$.



Pf of Prop: (i) Say $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Then

$$\|cv\| = \left\| \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix} \right\| = \sqrt{\sum (ca_i)^2} = \sqrt{c^2 \sum a_i^2} = |c| \cdot \sqrt{\sum a_i^2} = c\|v\|.$$

(ii). • Since v' is a positive mult. of v, v' has the same direction as v.

• $\|v'\| = \left\| \frac{1}{\|v\|} \cdot v \right\| = \frac{1}{\|v\|} \cdot \|v\| = 1.$

It follows that $\overset{c}{}$ v' is the normalization of v.

Eg. Normalize $v = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \in \mathbb{R}^3$.

Soln: $\|v\| = \sqrt{2^2+3^2+7^2} = \sqrt{62}$,

so the normalization of v is

$$v' = \frac{1}{\|v\|} \cdot v = \frac{1}{\sqrt{62}} \cdot \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{62} \\ 3/\sqrt{62} \\ 7/\sqrt{62} \end{bmatrix}.$$

Next time:

§ 6.2. Orthogonality