# Math 2135. Lecture 38.

12.01.2021.

**Last time:** · diagonalization exercises

· why $\overset{(n\times n)}{A} = PDP^{-1}$ when (*) A is diagonalizable, the cols of P form an eigenbasis (union of the bases of the e-spaces) and D is the diagonal matrix with the e-values listed w/ multiplicities:

for the map $T: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \mapsto A\vec{x}$, the e-basis $B$ and standard basis $C$ (of $\mathbb{R}^n$),

$$A = [T]_C^C, \quad D = [T]_B^B, \quad P = P_{C\leftarrow B} \quad (\text{so } P^{-1} = P_{B\leftarrow C})$$

**Today:** · more on "$D = [T]_B^B$"

· complex eigenvalues

· inner product

## 1. A closer look at "$D = [T]_\beta^\beta$"

$= \{v_1, \dots, v_n\}$

**Recall**: Given a linear map $T: V \to W$, a basis $\beta$ of $V$, and a basis $\gamma$ of $W$,

the matrix of $T$ w.r.t $\beta$ and $\gamma$ is $[T]_\beta^\gamma = \left[ [T(v_1)]_\gamma \middle| [T(v_2)]_\gamma \middle| \cdots \middle| [T(v_n)]_\gamma \right]$

So, in our setting (*) where $T(\vec{x}) = Ax$, $\beta$ is an eigenbasis of $A$,

say $Av_i = \lambda_i v_i \ \forall i$, then

$$[T]_\beta^\beta = \left[ [T(v_1)]_{v_1} \middle| \cdots \middle| [T(v_n)]_{v_n} \right] = \left[ [Av_1]_\beta \middle| \cdots \middle| [Av_n]_\beta \right]$$

$$= \left[ [\lambda_1 v_1]_\beta \middle| \cdots \middle| [\lambda_n v_n]_\beta \right] = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & \cdot & & \lambda_n \end{bmatrix} \to D.$$

Eg. (1) $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ is diagonalizable. $\lambda_1 = -7$, mult. 1, $B_1 = \{ \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{v_1} \}$

$\lambda_2 = 3$, mult 1, $B_2 = \{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{v_2} \}$

$Av_1 = -7 v_1$, $Av_2 = 3 v_2$, $B = \{v_1, v_2\}$

"$P$" $= [v_1 | v_2] = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ ← note that this is $P_{C \leftarrow B}$ where $C = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

$$[T]_B^B = \Big[ [T(v_1)]_B \quad [T(v_2)]_B \Big] = \Big[ [Av_1]_B \quad [Av_2]_B \Big] = \Big[ [-7v_1]_B \quad [3v_2]_B \Big] = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$$

↑ this is exactly $D$!

Upshot: To find a basis $B$ in which $[T]_B^B$ is a diagonal matrix is the same as to find a diagonalization of $A$.

**Eg.** $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ (from a previous lecture)

diagonalization (see before)

$\lambda_1 = 1, \quad B_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = -2, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Downarrow$

$A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

find a basis $\mathcal{B}$ of $\mathbb{R}^3$ s.t. for $T : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$,

$[T]_{\mathcal{B}}^{\mathcal{B}}$ is diag.

$\lambda_1 = 1, \quad B_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = -2, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Downarrow$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$[T]_{\mathcal{B}}^{\mathcal{B}} = \text{"}D\text{"} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

**E.g.** Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto A\vec{x}$ where $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ s.t. $[T]_{\mathcal{B}}^{\mathcal{B}}$ is a diagonal matrix;

determine $[T]_{\mathcal{B}}^{\mathcal{B}}$ in this case.

**Strategy:** diagonalize $A$. the eigenbasis $\{v_1, \dots, v_n\}$ suffices as $\mathcal{B}$,

and in that case $[T]_{\mathcal{B}}^{\mathcal{B}} = $ "$D$" $= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

**Soln:** E.x. diagonalize $A$

$\to$ $\lambda_1 = 1$, mult. 1, e-basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\to$ $\lambda_2 = 3$, mult 1, e-basis $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

$\Downarrow$

Conclusion: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a matrix for which $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal;

in fact, $[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

## 2. Complex eigenvalues

Consider the polynomial $x^2+1$. Recall that it has no real root, i.e.,

the equation $x^2+1=0$ has no real number solns since $r^2+1>0$ $\forall r \in \mathbb{R}$.

But $x^2+1$ has two complex roots: $x^2+1=0 \Leftrightarrow x^2=-1 \Leftrightarrow x=\pm i$

**Q:** What if the char. polynomial of a matrix $A$ has complex roots (and nonreal)? (e.g. $A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$)

$$\Downarrow$$

$$\chi(A)=x^2+1$$

**A:** Then $A$ cannot be diagonalized over the real numbers,

but may still a diagonalization if we allow complex numbers

as eigenvalues and entries of eigenvectors.

Eg. $A = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \rightarrow \chi(A) = x^2+1$

roots of $\chi(A)$, i.e. the (complex) eigenvalues: $\lambda_1 = i$, $\lambda_2 = -i$, both

w/ alg. mult = geom. mult = 1.

So $A$ is diagonalizable over the complex numbers

$$E_{\lambda_1} = \mathrm{Nul}(A - \lambda_1 I) = \mathrm{Nul}\left(\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \end{bmatrix} \begin{array}{c} \times(i) \\ \downarrow + \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 1+\underbrace{i^2}_{-1} & 0 \end{array}\right] = \begin{array}{ccc} x & y & \\ \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Rightarrow E_{\lambda_1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{l} x+iy=0 \\ y \in \mathbb{C} \end{array} \right\} = \left\{ \begin{bmatrix} -iy \\ y \end{bmatrix} = \underline{y\begin{bmatrix} -i \\ 1 \end{bmatrix}} : y \in \mathbb{C} \right\}.$$

p.v.f. $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

$$E_{\lambda_2} = \text{Null}(A - \lambda_2 I) = \text{Null}\left(\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right] \xrightarrow{\times(-i)\,+} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow E_{\lambda_2} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{c} x - iy = 0 \\ y \in \mathbb{C} \end{array} \right\} = \left\{ \begin{bmatrix} iy \\ y \end{bmatrix} = \underline{y \cdot \begin{bmatrix} i \\ 1 \end{bmatrix}} : y \in \mathbb{C} \right\}.$$

p.w.f. $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

So, over the complex numbers, $A$ is diagonalizable:

we have $A = PDP^{-1}$ where $P = \{v_1, v_2\} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. $\square$

## 3. Inner Products in $\mathbb{R}^n$ (starts Ch. 6.)

**Def.** (inner product : (vec, vec) $\mapsto$ scalar) The inner product or dot product of two vectors $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the scalar

$$u \cdot v = \langle u, v \rangle := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

as we've been using in matrix mult.

**Eg.** $n = 2$, $\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle = 1 \cdot 3 + 2 \cdot 4 = 11$

$n = 3$, $\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$

$n = 3$, $\left\langle \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = 2x + 3y + 5z$.

**Prop:** (properties of inner products). Let $u, v, w \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then

(1) $u \cdot v = v \cdot u$.

(2) $(u+v) \cdot w = u \cdot w + v \cdot w$

(3) $c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$

(4) $u \cdot u \geq 0$, with $u \cdot u = 0$ iff $u = \vec{0}$.

**Next time:**

- proofs of the properties
- applications / geometry of inner products.