Math 2135. Lecture 37. 11. 29. 2021.

Last time: · The diagonalization thm:

When and how we can diagonalize a matrix A $(\to A = PDP^{-1})$ — D diagonal

Step 1: find the eigenvalues of A by solving the char. equation $\chi(A) = 0$. $\to \lambda_1, \lambda_2, \cdots, \lambda_k$

Step 2: Check if $\underbrace{\text{alg. mult}(\lambda_i) = \text{geom. mult}(\lambda_i)}_{\text{automatically true if alg. mult}(\lambda_i) = 1.}$ $\forall\ 1 \le i \le k$.

true $\Rightarrow$ A is diagonalizable, P, D can be assembled from eigenbases and eigenvalues suitably.

false $\Rightarrow$ A is not diagonalizable.

Today: · more diagonalization exercises · explanation for "$A = PDP^{-1}$".

1. Diagonalization practice

1a). Determine if the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is

If so, find a closed formula for A^k $(k \in \mathbb{Z}_{>0})$.

Soln: $\chi(A) = \det \begin{bmatrix} 7-x & 2 \\ -4 & 1-x \end{bmatrix} = (7-x)(1-x) + 8 = 7 - 8x + x^2 + 8$

$= x^2 - 8x + 15 = (x-3)(x-5)$

So the e-values are $\lambda_1 = 3$, $\lambda_2 = 5$, both with alg mult, 1.

In particular, we have alg.mult (λ_i) = geom.mult(λ_i) $\forall i \in \{1,2\}$.

so A is diagonalizable.

To get the diag. $A = PDP^{-1}$, we need to compute the eigenbases for the eigenspaces E_3, E_5.

$\underline{E_3:}$

$$E_3 = \mathrm{Null}(A - 3I) = \mathrm{Null}\left(\begin{bmatrix} 7-3 & 2 \\ -4 & 1-3 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ -4 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{ccc} x & y & \\ \hline 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}$$

$$\Rightarrow E_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{l} x + y/2 = 0 \\ y \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} -y/2 \\ y \end{bmatrix} : y \in \mathbb{R} \right\} = \left\{ y \cdot \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}.$$

so $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$ is a basis for E_3.

$\underline{E_5:}$

$$E_5 = \mathrm{Null}\left(\begin{bmatrix} 7-5 & 2 \\ -4 & -5 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}\right)$$

$$\cdots\cdots \xrightarrow{\text{E.x.}} E_5 = \left\{ y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\},$$

so $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_5.

$\lambda_1 = 3.$ alg. mult (λ_1) = geom. mult $(\lambda_1) = 1.$ basis for E_3 : $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = 5$ .... (λ_2) = ... $(\lambda_2) = 1$, ... E_5 = $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$

So $A = PDP^{-1}$ where $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix}.$

(also, $A = P'D'P'^{-1}$ where $D' = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ and $P' = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix}$.)

$$A^k = (PDP^{-1})^k = PD\cancel{P^{-1} \cdot P}D\cancel{P^{-1} \cdot} \cdots \cancel{\cdot P}DP^{-1} = PD^kP^{-1}$$

$$= \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} \cdot \frac{1}{1/2} \cdot \begin{bmatrix} 1 & +1 \\ -1 & -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \cdot 3^k & 2 \cdot 3^k \\ -2 \cdot 5^k & -1 \cdot 5^k \end{bmatrix} = \begin{bmatrix} -3^k - 2 \cdot 5^k & -3^k - 5^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 1 \cdot 5^k \end{bmatrix}. \quad \square$$

(b) Repeat (a) for $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$.

Soln: $\chi(A) = \det \begin{bmatrix} -2-x & 12 \\ -1 & 5-x \end{bmatrix} = (-2-x)(5-x) + 12 = x^2 - 3x - 10 + 12$

$$= x^2 - 3x + 2 = (x-1)(x-2).$$

So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, both with alg. mult 1.

In particular, A must be diagonalizable.

Ex. compute e-bases for E_1 and E_2.

$$\to \quad A = PDP^{-1} \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; P = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}.$$

... compute A^k ... .

(c) Diagonalize the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible.

Soln: (sketch)

$$\chi(A) = \det \begin{bmatrix} 2-x & 4 & 3 \\ -4 & -6-x & -3 \\ 3 & 3 & 1-x \end{bmatrix} \overset{Ex}{=} \cdots = -(x-1)(x+2)^2$$

So the eigenvalues of A are $\lambda_1 = 1$ & $\lambda_2 = -2$, with alg. multiplicities 1 and 2, resp.

Diagonalizable?

λ_1 : alg. mult $(\lambda_1) = 1 \Rightarrow$ alg. mult $(\lambda_1) =$ geom. mult (λ_1).

λ_2 : $2 =$ alg. mult $(\lambda_1) \overset{?}{=}$ geom. mult. (λ_2)

geom. mult (λ_2) : $E_{\lambda_2} = \mathrm{Null}\left(\begin{bmatrix} 2-(-2) & 4 & 3 \\ -4 & -6-(-2) & -3 \\ 3 & 3 & 1-(-2) \end{bmatrix} \right)$

$$= \text{Null}\left(\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \to \begin{bmatrix} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

"y" is the only non-pivot variable, so

free

$$\text{geom. mult.}(\lambda_2) = \dim E_{\lambda_2} = 1 < \text{alg. mult}(\lambda_2)$$

So A cannot be diagonalized. $\square$

(d) Repeat (c) for $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$.

Soln: $\chi(A) = \det \begin{bmatrix} 5-x & -8 & 1 \\ 0 & 0-x & 7 \\ 0 & 0 & 2-x \end{bmatrix} = -x(x-2)(x-5)$

So the eigenvalues of A are 0, 2, and 5, all with alg. mult 1.

In particular, A must be diagonalizable.

Ex: diagonalize A.

2. Explanation for "$A = PDP^{-1}$"

Setting: A is diagonalizable, with eigenvalues $\lambda_1, \dots, \lambda_k$. Say A is $N \times N$.

Say has alg.mult (λ_i) = geom.mult $(\lambda_i) = n_i$ for some $n_i \in \mathbb{Z}$ $\forall i$.

say E has eigenbasis $\{v_{i1}, v_{i2}, \dots, v_{in_i}\} \longrightarrow B_i = [v_{i1} | \cdots | v_{in_i}]$

We will explain why $A = PDP^{-1}$ where $D = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{bmatrix}$ → size $n_1 \times n_1$, → size $n_2 \times n_2$

and $P = [B_1 | \cdots | B_k]$

To do so, we consider the standard basis $\{e_1, e_2, \dots, e_N\}$ of $\mathbb{R}^N$,

the basis $B := B_1 \cup B_2 \cup \cdots \cup B_k$ of eigenvectors,

and the linear map $T: \mathbb{R}^N \to \mathbb{R}^N$, $\vec{x} \mapsto A\vec{x}$.

Now, note that

- $P_{C \leftarrow B} = P$,

- $[T]_B^B = D$ and $[T]_C^C = A$ by the definition of matrices of linear maps w.r.t to chosen bases.

We also note (but will not prove) that

- $$[T]_C^C \overset{(*)}{=} P_{C \leftarrow B} [T]_B^B P_{B \leftarrow C} \quad \text{where } P_{B \leftarrow C} = P_{C \leftarrow B}^{-1}.$$

It follows that $$A = P \cdot D \cdot P^{-1}.$$

Next time:

- exercises related to $(*)$
- complex eigenvalues
- inner products