# Math 2135. Lecture 36

11. 19. 2021.

**Last time:** (Let $A$ be a square matrix throughout this lecture.)

To find the e-values & e-vects of $A$, use the facts that

- the e-values of $A$ are the roots of the char. eq. $\chi(A)$.
- the e-space of each e-value $\lambda$ is $E_\lambda = \mathrm{Null}(A - \lambda I)$.

in particular, we can find a basis and the dim of $E_\lambda$.

**Today:** how eigenvalues/eigenvectors relate to the diagonalization problem:

diagonal

$$A \longrightarrow A = PDP^{-1}$$

invertible

## 1. Algebraic vs. geometric multiplicity (of eigenvalues)

**Def 1.** Let $\lambda$ be an eigenvalue of $A$, so that $\chi(A) = \det(A - xI) = (x-\lambda)^m \cdots$.

Then the algebraic multiplicity of $\lambda$ is the largest positive number $m$ such that $(x-\lambda)^m$ divides $\chi(A)$, i.e., it is the multiplicity of the root $\lambda$ for $\chi(A)$.

**E.g.** By the last lecture the matrix $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ has char. poly

$\chi(B) = -(x-1)(x+2)^2$, so $B$ has two eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$;

they have algebraic mult. 1 and 2, respectively.

**Def 2.** Let $\lambda$ be an eigenvalue of $A$. Then the <u>geometric multiplicity</u> of $\lambda$ is the dimension of the eigenspace $E_\lambda$ of $A$.

**Eg.** Same $B$ as before: $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -2$.

$E_{\lambda_1}$: $E_{\lambda_1} = \mathrm{Null}(B - \lambda_1 I) = \mathrm{Null}\left(\begin{bmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow E_{\lambda_1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{c} x - z = 0 \\ y + z = 0 \\ z \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \cdot \underline{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} : z \in \mathbb{R} \right\}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of $E_{\lambda_1}$, so geom.mult$(\lambda_1) = \dim(E_{\lambda_1}) = 1$.

↘ equals alg.mult $(\lambda_1)$.

$\lambda_2 = -2$:  $E_{\lambda_2} = \mathrm{Null}(B - \lambda_2 I) = \mathrm{Null}\left(\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Rightarrow E_{\lambda_2} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{c} x+y+z=0 \\ y, z \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y \cdot \underline{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} + z \underline{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} : y, z \in \mathbb{R} \right\}.$$

$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $E_{\lambda_2}$, so

$$\text{geom. mult}(\lambda_2) = \dim(E_{\lambda_2}) = 2.$$

Note: again, geom. mult $(\lambda_2)$ = alg. mult. $(\lambda_2)$ here.

**E.g.** $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

**E.x:** show that $\chi(A) \overset{(*)}{=} (1-x)(x+2)^2$.

$$= -(x-\lambda_1)(x-\lambda_2)^2, \quad \lambda_1 = 1, \quad \lambda_2 = -2.$$

$$\Downarrow$$

$$\text{alg.mult}(\lambda_1) = 1, \quad \text{alg.mult}(\lambda_2) = 2.$$

**Ex.** $\text{geom.mult}(\lambda_1) = 1$.

We'll show that $\text{geom.mult}(\lambda_2) = 1 \leq \text{alg.mult}(\lambda_2)$:

$$E_{\lambda_2} = \text{Null}(A - \lambda_2 I) = \text{Null}\left(\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}\right).$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Rightarrow E_{\lambda_2} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x+y=0 \\ z=0 \\ y \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : y \in \mathbb{R} \right\} \Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } E_{\lambda_2},$$

so $\text{geom.mult}(\lambda_2) = 1$.

## 2. The diagonalizability theorem

Thm. Let $A$ be an $n \times n$ matrix.

(1). For every eigenvalue $\lambda$ of $A$, we have

$$1 \leq \text{geom. mult.}(\lambda) \leq \text{alg. mult}(\lambda)$$

(Note: as the last example shows, geom. mult $(\lambda)$ can be strictly small than alg. mult $(\lambda)$.)

(2). The matrix $A$ is diagonalizable, i.e., there exists a diagonal matrix $D$ and an invertible matrix $P$ s.t. $A = PDP^{-1}$, iff for every e-value $\lambda$ of $A$ we have alg. mult $(\lambda)$ = geom. mult $(\lambda)$.

(e.g. for the matrix $B$ earlier, this is the case, so $B$ is diagonalizable; for the matrix $A$ on the last page, this is not the case, so $A$ is not diagonalizable)

13) Suppose the condition shaded in green in 12) holds, i.e., suppose alg.mult $(\lambda)$ = geom.mult $(\lambda)$ for every e-value $\lambda$ of $A$. Suppose further that $B_1 = \{v_{11}, v_{12}, \dots, v_{1n_1}\}$, $B_2 = \{v_{21}, v_{22}, \dots, v_{2n_2}\}$, $\dots, B_k = \{v_{k1}, v_{k2}, \dots, v_{kn_k}\}$ are bases of the eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_k}$ where $\lambda_1, \dots, \lambda_k$ is a complete, irredundant list of e-values of $A$.

Then we have the set $B = \{v_{11}, \dots, v_{1n_1}, v_{21}, \dots, v_{2n_2}, \dots, v_{k1}, \dots, v_{kn_k}\}$ is a basis of $\mathbb{R}^n$, and

$$A = PDP^{-1} \text{ where } D = \left[\begin{array}{c|c|c} \underbrace{\begin{matrix}\lambda_1 & & \\ & \lambda_1 & \\ & & \ddots \; \lambda_1\end{matrix}}_{n_1\times n_1} & & \\ \hline & \underbrace{\begin{matrix}\lambda_2 & \\ & \ddots \; \lambda_2\end{matrix}}_{n_2\times n_2} & \\ \hline & & \ddots \end{array}\right] \text{ and } P = \left[\begin{array}{c|c|c} v_{11} \cdots v_{1n_1} & v_{21} \cdots v_{2n_2} & \cdots \end{array}\right].$$

Eg. For the matrix $B$, we have $B = PDP^{-1}$ where

$$D = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -2 & 0 \\ 0 & 0 & -2 \end{array}\right], \quad P = \left[\begin{array}{c|cc} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$$

basis for $E_1$ basis for $E_{-2}$.

Note. If for some eigenvalue $\lambda$ of $A$, $\text{alg. mult}(\lambda) = 1$, then part (1)

of the forces $\text{geom. mult}(\lambda) = 1 = \text{alg. mult}(\lambda)$.

Corollary. If all eigenvalues of $A$ have alg. mult. 1, then $A$ is diagonalizable.

In contrast, if some eigenvalue $\lambda$ of $A$ has $\text{alg. mult}(\lambda) > 1$, then to find out if $A$ is diagonalizable we generally have to compute $E_\lambda$ explicitly.

Eg. Determine if the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is diagonalizable.

If so, find a closed formula for $A^k$ where $k$ is a positive integer.

Next time:
- explaining the thm.
- diagonalization examples.
- inner products.