Math 2135. Lecture 35. 11.17.2017.

Last time:

- def. and motivation of diagonalization

  square matrix $A \xrightarrow[\text{find}]{\text{try to}}$ diagonal matrix D, inv. matrix P s.t. $A = PDP^{-1}$.

  useful for finding powers of A, and "better representing" lin. maps.

- def of eigenvalue and eigenvector of a square matrix A.

  A scalar $\lambda \in \mathbb{R}$ is an e-value of A if there is a nonzero vector $v \in \mathbb{R}^n$ s.t. $\underline{Av = \lambda v}$; any such vector v is called an e-vector of A w/ e-value λ.

Today.

- how to find eigenvalues and eigenvectors.

1. Finding eigenvalues

Let A be an $n\times n$ matrix.

Last time, we stated that the eigenvalues of A are precisely the roots of the characteristic polynomial $\chi(A) = \det(A - xI)$.

E.g. 1. Find all e-values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Soln: $\chi(A) = \det(A - xI) = \det\begin{bmatrix} 2-x & 3 \\ 3 & -6-x \end{bmatrix} = (2-x)(-6-x) - 9$

$$= x^2 + 4x - 12 - 9 = x^2 + 4x - 21$$

$x^2 + 4x - 21 = 0$

→ factor: $(x+7)(x-3) = 0 \Rightarrow x = -7$ or $x = 3$

→ use the quadratic formula: the solns are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$

$$= \frac{-4 \pm \sqrt{16+84}}{2} = -7 \text{ or } 3.$$

The e-values of A are -7 and 3.

Ex. Find all e-values of $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, assuming 1 is an e-value.

Soln: $\chi(A) = \det\left(\begin{bmatrix} 1-x & 3 & 3 \\ -3 & -5-x & -3 \\ 3 & 3 & 1-x \end{bmatrix}\right) = \left((1-x)(-5-x)(1-x) + (-27) + (-27)\right)$

$- 3\cdot 3\cdot(-5-x) + 9(1-x) + 9(1-x)$

$= -(+5+x)(x^2-2x+1) - 54 + 45 + 9x + 9 - 9x + 9 - 9x$

$= -(x^3 - 2x^2 + x + 5x^2 - 10x + 5) + 9 - 9x$

$= -x^3 - 3x^2 + 9x - 5 - 9x = -x^3 - 3x^2 + 4$

1 is an e-value $\Rightarrow$ $-x^3 - 3x^2 + 4 = (x-1)\cdot ?$ $\longrightarrow$ $= \boxed{(-x^3 + x^2) - x^2} - 3x^2 + 4$

$= -x^2(x-1) - 4x^2 + 4 = -x^2(x-1) - 4x^2 + 4x - 4x + 4$

$= -x^2(x-1) - 4x(x-1) - 4(x-1) = (x-1)(-x^2-4x-4) = \boxed{-(x-1)(x+2)^2}$

So the e-values of A are 1 and -2. ∎

2. Theory: e-values/e-vecs via $A - xI$

Let A be an $n \times n$ matrix.

We'll now justify why the e-values of A are the roots of $\det(A - xI)$ and explain how to find eigenvectors for a e-value λ.

Key observation: we have $Av = \lambda v \overset{=}{} (\lambda I_n) \cdot v$ for a nonzero vec $v \in \mathbb{R}^n$ and scalar $\lambda \in \mathbb{R}$

$\Leftrightarrow$ $Av - \lambda v = (A - \lambda I_n) v = 0$ for a nonzero vec $v \in \mathbb{R}^n$.

$\Leftrightarrow$ $\mathrm{Nul}(A - \lambda I_n)$ is nontrivial and $v \in \mathrm{Nul}(A - \lambda I_n)$

$\Leftrightarrow$ $\det(A - \lambda I_n) = 0$ and $v \in \mathrm{Nul}(A - \lambda I_n)$

Conclusion: (1) λ is e-value of A $\Leftrightarrow$ λ is a root of $\det(A - xI_n)$.

(2) If λ is an e-value of A, the eigenvectors of A w/ evalue λ forms precisely the set $\mathrm{Nul}(A - \lambda I_n)$.

3. Finding eigenvectors

Let A be an $n \times n$ matrix.

Def. (eigenspace) For each e-value λ of A, we call the set

$$E_\lambda := \{ v \in \mathbb{R}^n \mid Av = \lambda v \} \quad (= \mathrm{Null}(A - \lambda I_n))$$

the eigenspace of A w/. e-value λ.

E.g. Find all e-spaces of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. (recall that A has e-values -7 and 3)

Soln: $\lambda = -7$:

$$E_{-7} = \mathrm{Null}(A - (-7) I_2) = \mathrm{Null}\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} \right)$$

$$= \mathrm{Null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \right)$$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \to$$

$$\mathrm{Null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \begin{array}{c} x + y/3 = 0 \\ y \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} -y/3 \\ y \end{bmatrix} : y \in \mathbb{R} \right\} = \left\{ y \cdot \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}$$

$\lambda = 3$: $E_3 = \text{Null}(A - 3I_2) = \text{Null}\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \longrightarrow E_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{l} x - 3y = 0 \\ y \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} 3y \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$$

$$= \left\{ y \begin{bmatrix} 3 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}.$$

So the eigenspaces of A are $E_{-7} = \left\{ y \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}$

and $E_3 = \left\{ y \begin{bmatrix} 3 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}$.

Check: $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} +7/3 \\ -7 \end{bmatrix} = -7 \cdot \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. ✓

E.g. Find all e-values and e-spaces of $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

(Recall that $\chi(B) = -(x-1)(x+2)^2$)

Soln: Ex. find E_1 and E_{-2}. Check that $\dim E_1 = 1$, $\dim E_{-2} = 2$.

Next time: · algebraic vs. geometric multiplicities.

· the diagonalizability theorem.