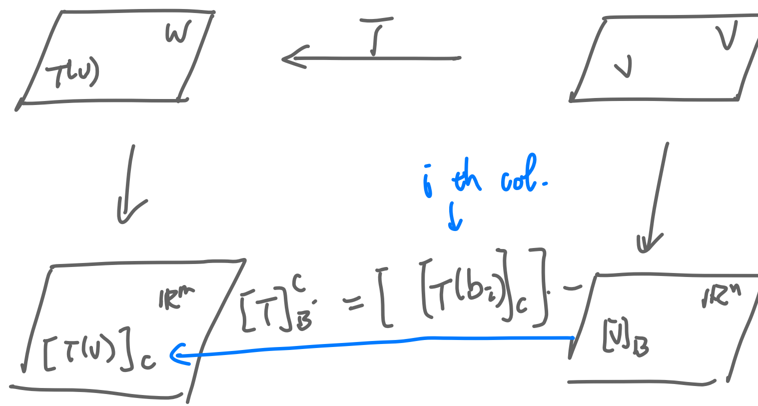
# Math 2135. Lecture 34.

11. 15. 2021.

**Last time:** · mentioned the matrix $[T]_B^C$ of a linear map $T: V \to W$ w.r.t. a basis $B = \{b_1, b_2, \dots, b_n\}$ of $V$ and a basis $C$ of $W$.
(say $|C| = m$)




**Today:** · Start Ch5. Diagonalization: motivation and strategy.

## 1. Two motivating examples

$$\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1/10 \cdot (-7)^k & 3/10 \cdot (-7)^k \\ 3/10 \cdot 3^k & 1/10 \cdot 3^k \end{bmatrix}$$

### (a) Computing matrix powers

Eg. Find a closed formula for $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}^k$ where $k$ is an arbitrary pos. int. $\longrightarrow$ hard ...

Useful Fact:
$$\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}}_{D,\ \text{diagonal}} \underbrace{\begin{bmatrix} -1/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}}_{P^{-1}}$$

Thus. $A^k = (PDP^{-1})^k = PD\cancel{P^{-1}P}D\cancel{P^{-1}P}DP^{-1} \cdots \cancel{P}DP^{-1}$ ($k$ copies)

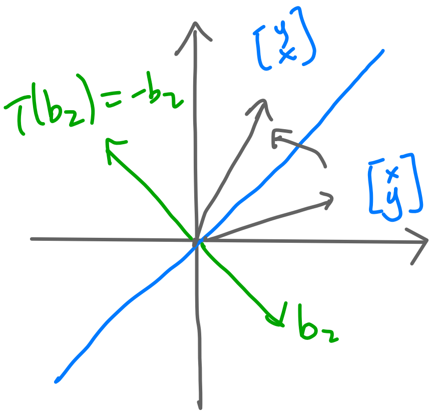
$$= P\underbrace{D^k}_{\text{easy}}P^{-1} = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} (-7)^k & 0 \\ 0 & 3^k \end{bmatrix} \cdot \begin{bmatrix} -1/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} (-7)^k + 3^{k+2} & .. \\ -3\cdot(-7)^k + 3^{k+1} & .. \end{bmatrix}$$

Point: Having $A$ in the form $A = PDP^{-1}$ for $D$ diagonal and $P$ inv. is helpful for computing $A^k$.

## (b) "Nice" matrix representations of linear maps

E.g. Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by refl. w.r.t the line $y = x$.

Consider the two bases $B_1 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2} \right\}$ $B_2 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b_2} \right\}$ of $\mathbb{R}^2$.




Then $T$ is linear, and

$$[T]^{B_1}_{B_1} = \left[ [T(e_1)]_{B_1} \,\middle|\, [T(e_2)]_{B_1} \right] = \left[ \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_{B_1} \,\middle|\, \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{B_1} \right]$$

Fact: if $B$ is the standard basis then $[T]^B_B$ is the usual standard matrix.

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{standard matrix}$$

$$[T]^{B_2}_{B_2} = \left[ [T(b_1)]_{B_2} \,\middle|\, [T(b_2)]_{B_2} \right] = \left[ \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{B_2} \,\middle|\, \left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{B_2} \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leftarrow \text{considered "nicer" since it's diagonal}$$

Both the above examples are related to the problem of diagonalizing square matrices (We'll explain the relationship in more details later).

**Def:** The <u>diagonalization</u> of a square matrix $A$ refers to the process of trying to find a diagonal matrix $D$ and an invertible matrix $P$ s.t. $A = PDP^{-1}$.

**Fact:** Such a pair $(D, P)$ does not always exist. If it does, we also refer to the decomp. $A \overset{*}{=} PDP^{-1}$ as a diagonalization of $A$.

e.g. $$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}$$

To get (*), we'll try to get the cols of $P$ & the diag entries of $D$.

- We'll get the diagonal entries of $D$ as "eigenvalues of $A$".
- We'll get the cols of $P$ as "eigenvectors of $A$".

## 2. Def of eigenvalues and eigenvectors

**Def:** Let $A$ be an $n\times n$ matrix. A scalar $\lambda \in \mathbb{R}$ is called an <u>eigenvalue</u> of $A$ if there is a nonzero vector $v \in \mathbb{R}^n$ s.t. $Av = \lambda v$. In this case, i.e., if $\lambda$ is an eigenvalue of $A$, then any vector $w$ s.t. $Aw = \lambda w$ is called an <u>eigenvector</u> of $A$ w/ eigenvalue $\lambda$.

**Eg.** Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Av = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\cdot v, \quad Aw = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\cdot w, \quad Au = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So $v$ is an e-vec of $A$ w/ e-value 3, $w$ is an e-vect of $A$ w/ e-value 0, $u$ is not an e-vec of $A$. In particular, 3 and 0 are e-values of $A$.

## 3. Finding eigenvalues via characteristic polynomials

**Def.** (characteristic polynomial) For every $n \times n$ matrix $A$, the <u>characteristic polynomial</u> of $A$ is the polynomial (in $x$)

$$\chi(A) = \det(A - x I_n).$$

**E.g.**

$$\chi\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-x & 1 \\ 2 & 2-x \end{bmatrix}\right)$$

$$= (1-x)(2-x) - 1 \cdot 2 = x^2 - 3x + 2 - 2 = x^2 - 3x$$

<u>Note</u>, $(x^2 - 3x) = x(x-3)$, so the roots of the char. poly. (i.e., the solns of $\chi(A) = 0$) are $0$ and $3$.

Why is the char. poly. important?

**Thm 1.** The eigenvalues of $A$ are precisely the roots of the characteristic poly for any square matrix $A$!

**Ex. 1.** Find all e-values of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

**Ex. 2.** Find all e-values of $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, assuming $1$ is an e-value.

Next time:
- Ex. 2.
- proof of Thm 1.
- finding eigenvectors.