Math 2135. Lecture 33. 11.10.2021.

Last time: change-of-basis matrices $P_{C\leftarrow B}$.

- Context: B, C are two bases of a vec. space V, $B=\{b_1, b_2, \cdots, b_n\}$, $C=\{c_1, \cdots, c_n\}$,
- what $P_{C\leftarrow B}$ is: $P_{C\leftarrow B} = \left[[b_1]_C \,\middle|\, [b_2]_C \,\middle|\, \cdots \,\middle|\, [b_n]_C \right]$
- what $P_{C\leftarrow B}$ does (property): $\forall x \in V$, $[x]_C = P_{C\leftarrow B}[x]_B$
- reciprocity: $P_{B\leftarrow C} = \left(P_{C\leftarrow B}\right)^{-1}$

Today:

- how to compute $P_{C\leftarrow B}$ efficiently.
- a more general setting: the matrix of a linear map w.r.t. two bases

(not on the midterm)

1. Algorithm for finding $P_{C\leftarrow B}$

Let B, C, V be as before. Write $M_B = [b_1 | b_2 | \cdots | b_n]$, $M_C = [c_1 | \cdots | c_n]$.

E.g. Find $P_{C\leftarrow B}$ for the bases $B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

Soln: We need $[b_1]_C$ and $[b_2]_C$.

$[b_1]_C$: solving $x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $\longrightarrow \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\longrightarrow \left[\begin{array}{cc|c} 2 & 3 & -2 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{ccc} x & y & \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & 1 \end{array}\right] \Longrightarrow \vec{x} = [b_1]_C = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$$

$\underbrace{}_{M_C}$ b_1 $\qquad\qquad$ $\underbrace{}_{I_2}$ $M_C^{-1} \cdot b_1$

same as left mult by M_C^{-1}.

$[b_2]_C$: $\quad x\begin{bmatrix}2\\0\end{bmatrix} + y\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 1 & 1 \end{array}\right] \\ \underbrace{}_{M_C}\;\; b_2 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & -3/2 \\ 0 & 1 & 1 \end{array}\right] \\ \underbrace{}_{I_2}\;\; [b_2]_C \end{array} \Rightarrow [b_2]_C = \begin{bmatrix}-3/2\\1\end{bmatrix}.$$

$[b_2]_C = \vec{x} = M_C^{-1} \cdot b_2$

$M_C \cdot \vec{x} = b_2$

same as $M_C^{-1} \cdot$, left mult. by M_C^{-1}

So $\quad P_{C \leftarrow B} = \left[[b_1]_C \;\; [b_2]_C \right] = \begin{bmatrix} -5/2 & -3/2 \\ 1 & 1 \end{bmatrix}$.

Note: We can combine the inversion of M_C in these two steps to speed up the computation. More precisely, ...

Thm. (Algorithm for finding $P_{C\leftarrow B}$) To find $P_{C\leftarrow B}$, it suffices to row reduce the matrix

$$[M_C \mid M_B] = [c_1 | \cdots | c_n \mid b_1 | \cdots | b_n]$$

until the left half becomes I_n. At this point, the right half of the resulting matrix will be $P_{C\leftarrow B}$. (In other words, $P_{C\leftarrow B} = M_C^{-1} M_B$.)

Eg. $V = \mathbb{R}^2$, $B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ $\leftarrow$ same example as before

Find $P_{C \leftarrow B}$.

Soln (in new form)

$$[M_C \mid M_B] = \left[\begin{array}{cc|cc} 2 & 3 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\underbrace{\hphantom{2\ 3}}_{M_C} \quad b_1 \quad b_2$$

$$\to \left[\begin{array}{cc|cc} 1 & 3/2 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & -3/2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\underbrace{\hphantom{1\ 0}}_{I_2} \quad [b_1]_C \quad [b_2]_C$$

So $P_{C \leftarrow B} = \begin{bmatrix} -5/2 & -3/2 \\ 1 & 1 \end{bmatrix}$.

Ex. Use the algorithm to find $P_{B \leftarrow C}$, then check that $P_{C \leftarrow B} = P_{B \leftarrow C}^{-1}$.

E.g. $V = \mathbb{R}^2$, $C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$, $B = \left\{ \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right\}$. Find $P_{C \leftarrow B}$.

Soln:

$$\left[M_C \mid M_B \right] = \left[\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right].$$

It follows that

$$P_{C \leftarrow B} = \begin{array}{cc} \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \\ [b_1]_C \; [b_2]_C \end{array}. \quad \square$$

Check:

$$-1 \cdot c_1 + 2 \cdot c_2 = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad -2 \cdot c_1 + 3 \cdot c_2 = -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

b_1 ✓ $\qquad$ b_2 ✓

More examples in HW.

2. Matrix of a linear map

say $B = \{b_1, \dots, b_n\}$

Setting: Let V, W be vector spaces, and let $T: V \to W$ be a linear map.

Let B be a basis of V; let C be a basis of W. Say $|B| = n$, $|C| = m$.

Thm: There is a unique matrix M ($m \times n$, with real numbers as entries) s.t.

$$\forall x \in V, \qquad \underbrace{[\underset{\in W}{\underline{T(x)}}]_C}_{\in \mathbb{R}^m} = M \cdot \underbrace{[\underset{\in V}{\underline{x}}]_B}_{\in \mathbb{R}^n}.$$

Moreover, the matrix M is given by
$$M = \Big[[T(b_1)]_C \,\Big|\, \cdots \,\Big|\, [T(b_n)]_C \Big].$$

In a picture

$$\begin{tikzcd} W \ni T(x) \arrow[d, "[\ ]_C"'] & V \ni x \arrow[l, "T"'] \arrow[d, "[\ ]_B"] \\ \mathbb{R}^m \ni [T(x)]_C & \mathbb{R}^n \ni [x]_B \arrow[l, "M\cdot"'] \end{tikzcd}$$

~~Def~~ Notation: We'll denote the matrix M by $[T]_B^C$, and we call M the matrix of T w.r.t. B and C.

Note: Everything is done relative to the chosen bases B, C.

E.g. Suppose V has a basis $B = \{b_1, b_2\}$ and W has a basis $C = \{c_1, c_2, c_3\}$.

Suppose that $T: V \to W$ is the lin. map s.t.

$$T(b_1) = 3c_1 - 2c_2 + 5c_3, \quad T(b_2) = 4c_1 + 7c_2 - c_3.$$

Then $$[T]_B^C = \left[[T(b_1)]_C \,\middle|\, [T(b_2)]_C \right] = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}.$$

E.g. Let $T: P_2 \to P_1$ be "formal differentiation". Then w.r.t to the standard bases $B = \{1, t, t^2\}$ and $C = \{1, t\}$ of P_2 and P_1, respectively,

$$T(1) = 0, \quad T(t) = 1, \quad T(t^2) = 2t = 2 \cdot t$$

So $$[T]_B^C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

More examples after the midterm.

E.g. (Connection to change-of-basis)

When $V = W$ and $T = \mathrm{Id}_V$, the picture

$$\begin{tikzcd}
V \ni T(x) = x \arrow[d, "{[\ ]_C}"'] & V \ni x \arrow[l, "T = \mathrm{Id}"'] \arrow[d, "{[\ ]_B}"] \\
\mathbb{R}^n \ni [x]_C & \mathbb{R}^n \ni [x]_B \arrow[l, "M"']
\end{tikzcd}$$

recovers

$$\begin{tikzcd}
 & V \ni x \arrow[dl, "{[\ ]_C}"'] \arrow[dr, "{[\ ]_B}"] & \\
\mathbb{R}^n \ni [x]_C & & \mathbb{R}^n \ni [x]_B \arrow[ll, "P_{C \leftarrow B}"']
\end{tikzcd}$$

,

so $[\mathrm{Id}]_B^C = P_{C \leftarrow B}$ by the uniqueness part of our theorem.