Math 2135. Lecture 32.

11.08.2021.

Last time: HW, proof worksheet, Midterm review.

Today: · Change of basis. (4.7)

Setting: V: a vec. space, B, C: two bases of V, x: an elt of V.

$\longrightarrow$ coord. vectors

$[x]_B$ how x decomposes w.r.t. B

$[x]_C$ - - - - - - - - - - C.

Q: How are $[x]_B$ and $[x]_C$ related?

A: There is a unique matrix P s.t. $[x]_C = P\,[x]_c$.

In a picture:

$$
\begin{tikzcd}
& x \in V \arrow[dl, "{[\ \ ]_C}"'] \arrow[dr, "{[v]_B}"] & \\
{[x]_C \in \mathbb{R}^n} & & {[x]_B \in \mathbb{R}^n} \arrow[ll, "?"', "P."]
\end{tikzcd}
$$

, say $\dim V = n$

To visualize the problem, consider the two coordinate systems in Figure 1. In Figure 1(a), $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, while in Figure 1(b), the same $\mathbf{x}$ is shown as $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$. That is,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Our problem is to find the connection between the two coordinate vectors. Example 1 shows how to do this, provided we know how $\mathbf{b}_1$ and $\mathbf{b}_2$ are formed from $\mathbf{c}_1$ and $\mathbf{c}_2$.

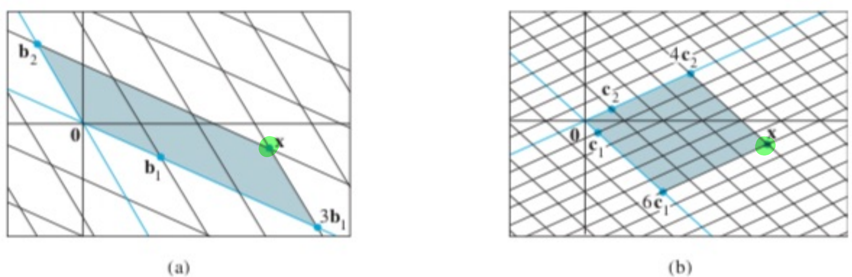

FIGURE 1 Two coordinate systems for the same vector space.

Eg: Let V be a v.s with dim 2 and bases $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$

s.t $b_1 = 4c_1 + c_2$, $b_2 = -6c_1 + c_2$. $\rightarrow$ so $[b_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $[b_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

Consider $x \overset{*}{=} 3b_1 + b_2$. so that $[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

What is $[x]_C$? $\underline{[x]_C} \overset{*}{=} [3b_1 + b_2]_C$

$= 3[b_1]_C + [b_2]_C$ since $[\ ]_C$ is linear

$= 3 \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}}_{P = [[b_1]_C\ [b_2]_C]} \cdot \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[x]_B}$

Point: If we know how to decompose x w.r.t B and how each elt in B decomposes w.r.t. C, then we should know how to decompose x w.r.t. C.

More precisely, ...

Thm: Let V be a vec. space and let $\mathcal{B} = \{b_1, \dots, b_n\}$, $\mathcal{C} = \{c_1, \dots, c_n\}$ be two bases of V. Then there is a unique matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t.

$$\forall x \in V, \quad \text{we have} \quad [x]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}.$$

This matrix is given by

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \end{bmatrix}.$$

Def. In the above setting, the matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is called the <u>change-of-basis matrix</u> from $\mathcal{B}$ to $\mathcal{C}$.

One more example:

$V = \mathbb{R}^2$, $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. $B = \left\{ \underset{b_1}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} \right\}$, $C = \left\{ \underset{c_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{c_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$

Note that → the four observations covered in require computations in general cases!

- (nontrivial observation) $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = -2 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, so $[v]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

- (- - - - - - .) $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $[v]_C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

- $P_{C \leftarrow B} = \left[[b_1]_C \,\middle|\, [b_2]_C \right] = \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix}$.

$[b_1]_C$: $\underset{b_1}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} = -1 \cdot \underset{c_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + 0 \cdot \underset{c_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$, so $[b_1]_C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$[b_2]_C$: $\begin{bmatrix} 3 \\ 5 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $[b_2]_C = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- Check: $P_{C \leftarrow B} \cdot [v]_B = \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [v]_C$, as the thm. promises!

Pf of the theorem : (the matrix $P = \big[[b_1]_C \cdots [b_n]_C \big]$ is the unique matrix s.t.

$$[x]_C = P\,[x]_B \quad (*) \Big)$$

Two parts :

(1) $*$ holds. Let $x \in V$. Say $[x]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$, so $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$.

Then
$$[x]_C = [\alpha_1 b_1 + \cdots + \alpha_n b_n]_C$$
$$= \underbrace{\alpha_1}_{\in \mathbb{R}} \underbrace{[b_1]_C}_{\text{vec}} + \cdots + \underbrace{\alpha_n}_{\in \mathbb{R}} \underbrace{[b_n]_C}_{\text{vec}}$$
$$= \big[[b_1]_C \cdots [b_n]_C \big] \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{[x]_B}.$$
$$= P \cdot [x]_B.$$

(2) uniqueness : if M is an $n \times n$ matrix s.t. $[x]_C = M \cdot [x]_B \; \forall x \in V$, then $M = P$. $\rightarrow$ Ex: Consider $x = b_1$, $x = b_2$, $\cdots$, $x = b_n$.

A corollary: In the setting of the thm,

$$P_{B \leftarrow C} = \left(P_{C \leftarrow B} \right)^{-1}.$$

Pf: For all $x \in V$, we have

$$[x]_C = P_{C \leftarrow B} [x]_B ,$$

So $$\left(P_{C \leftarrow B}^{-1} \right) \cdot [x]_C = [x]_B \quad \forall x \in V.$$

By the uniqueness part of the theorem, it follows that $P_{C \leftarrow B}^{-1}$ is the change-of-basis matrix from C to B. i.e., $P_{C \leftarrow B}^{-1} = P_{B \leftarrow C}$.

Ex: Compute $P_{B \leftarrow C} = [[c_1]_B \ [c_2]_B]$ in the last example and verify that its inverse to $P_{C \leftarrow B}$.

Exercise: Suppose that $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are two bases of a vec. space V. Suppose

$$a_1 = 4b_1 - b_2, \quad a_2 = -b_1 + b_2 + b_3, \quad a_3 = b_2 - 2b_3.$$

(i) Find $P_{A \leftarrow B}$ and $P_{B \leftarrow A}$.

(ii) Find $[x]_B$ for $x = 3a_1 + 4a_2 + a_3$.

Next time:

- computing $P_{C \leftarrow B}$.
- generalizing the thm.

Soln: (i) $P_{B \leftarrow A} = \begin{bmatrix} [a_1]_B & [a_2]_B & [a_3]_B \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$

It follows that $P_{A \leftarrow B} = P_{B \leftarrow A}^{-1} = \ldots$ (Ex. use the inv. algorithm).

(ii) By assumption, $[x]_A = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$. Thus, by the theorem,

$$[x]_B = P_{B \leftarrow A}[x]_A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}. \qquad \square$$