Math 2135. Lecture 30.

So far: • "old to new": lin. comb, span, lin ind., basis, lin. maps, subspaces all make sense not only for $\mathbb{R}^n$ but also for abstract vector spaces. Similarly, we have the notions of dimension, injectivity, surjectivity, etc ...

• Coordinate mappings: given a v.s. V and any basis $\mathcal{B}$ of V, we have a map $[\ ]_{\mathcal{B}} : V \to \mathbb{R}^p$ where $p = |\mathcal{B}|$, $v \mapsto [v]_{\mathcal{B}}$.

Today: • new to old: solving problems about new v.s. using $\mathbb{R}^n$ $(n \in \mathbb{Z})$ and coordinate mappings.

• review of 4.1–4.6

1. More on coordinate mappings.

Setup: Let V be an abstract vec. space, let B be a basis of V. $= \{v_1, \dots, v_n\}$

Suppose $|B| = n$. (so $\dim V = n$)

Prop: The coordinate map $[\ ]_B : V \to \mathbb{R}^n$, $v \mapsto [v]_B$ is a bijective linear map.

Pf: (1) " $[v+w]_B = [v]_B + [w]_B$ ". Let $v, w \in V$. Suppose $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $[w]_B = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

Then $v = c_1 v_1 + \cdots + c_n v_n$, $w = d_1 v_1 + \cdots + d_n v_n$. So

$$[v+w]_B = [(c_1 v_1 + \cdots + c_n v_n) + (d_1 v_1 + \cdots + d_n v_n)]_B$$

$$= [(c_1+d_1)v_1 + \cdots + (c_n+d_n)v_n]_B = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} = [v]_B + [w]_B.$$

That is, $[\ ]_B$ respects addition.

(2) "$[cv]_B = c[v]_B$". Let $c \in \mathbb{R}$, $v \in V$. Suppose $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

then $v = c_1 v_1 + \cdots + c_n v_n$, so

$$[cv]_B = [c(c_1 v_1 + \cdots + c_n v_n)]_B = [(cc_1)v_1 + \cdots + (cc_n)v_n]_B$$
$$= \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c[v]_B.$$

That is, $[\ ]_B$ respects scaling.

By (1) and (2), $[\ ]_B$ is linear.

(3) Why is $[\ ]_B$ surjective? For any elt $y = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$, we have

$$[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{for } v = a_1 v_1 + \cdots + a_n v_n,$$

so y is an output of $[\ ]_B$. Therefore $[\ ]_B$ is surj.

(4) Why is $[\ ]_B$ injective?

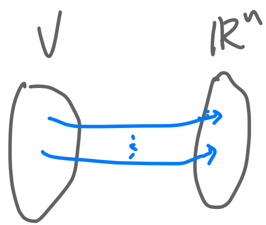
Suppose $v, w \in V$ satisfies $[v]_B = [w]_B$, say $[v]_B = [w]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Then $v = w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$.

In particular, $v = w$, so $[\ ]_B$ is inj.

By (1)-(4), $[\ ]_B$ is a bijective linear map.

Point:



The bijective linear map (called an "isomorphism") sets up a "dictionary" between (the elts) of V & $\mathbb{R}^n$.

Fact: In the above setting $(V, B, [\ ]_B)$. Let $S = \{v_1, \ldots, v_n\} \subseteq V$. Then S often satisfies a property in V iff the set $\{[v_1]_B, \ldots, [v_n]_B\}$ satisfies the same properties in $\mathbb{R}^n$.

(that can be phrased in terms of lin. alg. eg. lin. ind.)

Eg: $V = P_2$. Is the set $\{\underset{v_1}{1-t}, \underset{v_2}{t^2}, \underset{v_3}{3t}\}$ a basis of V?

Key: Here and often elsewhere, we need a "nice" basis $\mathcal{B}$ to translate the problem to one for $\mathbb{R}^n$.

Soln: Recall that P_2 has a basis $\mathcal{B} = \{1, t, t^2\}$.

Now, $[v_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $[v_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[v_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The vectors $[v_1]_{\mathcal{B}}, [v_2]_{\mathcal{B}}, [v_3]_{\mathcal{B}}$ form a basis of $\mathbb{R}^3$ since the above EF has no row of zeros. therefore v_1, v_2, v_3 form a basis of V. $\square$

Eg. The set $B = \{ \underset{v_1}{1 + t^2},\ \underset{v_2}{t + t^2},\ \underset{v_3}{1 + 2t + t^2} \}$ is a basis for P_2.

Find the coordinate vector of $v = 1 + 4t + 7t^2$ relative to B.

The problem: to decompose v in terms of v_1, v_2, v_3, i.e., to find

x_1, x_2, x_3 s.t. $x_1 v_1 + x_2 v_2 + x_3 v_3 = v$

An equivalent problem: to decompose $[v]_B$ in terms of $[v_1]_B, [v_2]_B, [v_3]_B$ for the basis $B = \{1, t, t^2\}$, i.e., to find x_1, x_2, x_3 s.t.

$$x_1 [v_1]_B + x_2 [v_2]_B + x_3 [v_3]_B = [v]_B.$$

$$\iff \quad x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}.$$

Ex: Complete the problem.

2. Review of 4.1 – 4.6

→ Read the sections.

Next time:
- problems from 4.1 – 4.6.
- the proof worksheet.

4.1. Vector spaces and subspaces

Key: definitions. the three properties for subspaces.

4.2. Null spaces, Column spaces, and linear transformations

for matrices. same as in Ch. 2.

4.3. Linearly independence sets; Bases.

all part of our "old to new" generalizations.

Know the definitions!

4.4. Coordinate systems → today.

4.5. The dimension of a vector space.

4.6. Rank → for matrices, as in Ch 2.

they are the same, namely # pivots in A.

row rank of a matrix $A \stackrel{\text{def}}{=} \dim(\mathrm{Row}(A))$

Col rank of . . . . . $\stackrel{\text{def}}{=} \dim(\mathrm{Col}(A))$