Math 2135. Lecture 3.

08/27/2021

Last time: · Def. of elementary row operations:

E1. interchange E2. scaling E3. replacement

· Def. of echelon and reduced echelon forms:

EF: (1) zero rows below nonzero rows (2)/(3): "staircase" condition

REF: (1) + (2)/(3) + (a) row leading entries are all 1 + (b) entries above row leading entries are zero.

· **Prop:** Using elt. row operations, we can transform every matrix A to an echelon form and a unique reduced echelon form.

Today. how and why to make the transformation.

1. **Gaussian elimination**: algorithm to transform a matrix to (reduced) echelon form using elt. row ops.

The (recursive) algorithm:

Step 1. Find the leftmost nonzero column. Interchanging rows if necessary, make sure the top entry in that column is nonzero. (e.g. $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \end{bmatrix}$)

Step 2. Use E3 to create zeros below the top nonzero entry from step 1.

running example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow[-5R_1]{R2 \leftarrow R2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & (6-2\cdot 5) & (7-3\cdot 5) & (8-4\cdot 5) \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{R3 \leftarrow R3 - 9R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

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Step 3. Repeat steps (1) and (2) on the submatrix to the lower right of the nonzero entry from Step (1). ↗ this entry is row-leading and called a pivot.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{(1)\ \&\ (2)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \end{array} \begin{array}{l} \\ R_x \\ R_y \end{array} \xrightarrow{R_y \leftarrow R_y - 2R_x} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{all zeros, halt!}$$

Repeat this process until the resulting submatrix is empty or contains only zeros.

This results in an echelon form.

Step 4. (for reduced echelon form)

To get the resulting echelon form step (3) into a reduced echelon form, use E2 to make sure every row leading entry is 1 and then use E3 to make sure entries above row. leading entries are zero. $\square$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E2} \underset{(a)}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \xrightarrow[R1 \leftarrow R1 - 2R2]{E3} \underset{(b)!}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

DONE!

E.g.

(a) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \end{bmatrix} \longrightarrow \underset{\text{E.F.}}{\begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \underset{\text{R.E.F.}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}}$

(b) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

E.F. and actually REF.

(c). $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \underset{\text{E.F.}}{\begin{bmatrix} 2 & 6 & 10 \\ 0 & 3 & 7 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \xrightarrow[R1-3R2]{R1 \uparrow} \underset{\text{R.E.F.}}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{7}{3} \end{bmatrix}}$

(d) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \underset{\text{E.F.}}{\begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -4 \end{bmatrix}} \longrightarrow \underset{\text{R.E.F.}}{\begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix}}$

2. Solns of L.E.S. from Echelon forms

E.g. (2). $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

Gaussian elimination

$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} x - z = -2 \\ y+2z = 3 \end{cases}$

$0x+0y+0z=0$. Duh!

inf. many solns.

next time: get this into "parametric vector form".

$$\text{soln set} = \left\{ (x,y,z) \;\middle|\; \begin{matrix} x = z-2 \\ y = 3-2z \end{matrix},\ z \in \mathbb{R} \right\} = \left\{ (z-2,\ 3-2z,\ z) \;\middle|\; z \in \mathbb{R} \right\}.$$

(2).

$$(*)\begin{cases} \quad\quad 3z = 9 \\ 2x \quad - z = 5 \\ \quad 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} =: A.$$

Ex: Find a reduced ech. form of A. Then find the soln set of $(*)$.

Next time: more on finding solns of lin. systems from (reduced) echelon forms.