Math 2135. Lecture 29.

11.01.2021.

Last time:

- using vector space axioms.
- familiar notions / facts from $\mathbb{R}^n$.
  - lin. comb. ← possible because we have $+$ and $\cdot$.
  - $\downarrow$
  - spans
  - subspace.

Def: A subset W of a v.s. V $(V, +, \cdot)$ is called a subspace if (a) ..., (b) ..., (c) ....

Point: W is a subspace $\Leftrightarrow$ $(W, +, \cdot)$ is a v.s. itself.

Today: · more generalized notions / facts. Let V be a real vector space.

1. Spans are subspaces.

Prop: Let $S \subseteq V$. Then the span of S is always a subspace of V.

Eg.: The set $W = \{ a + at - bt^3 : a, b \in \mathbb{R} \} \subseteq P_3$ is a subspace of P_3 because $W = \{ a(1+t) + b(-t^3) \} = \text{Span}\{ 1+t, -t^3 \}$.

Pf: We need to check that $\text{Span}(S)$ satisfies the three necessary properties. See "proof practice" worksheet.

2. Bases and dimension.

Def. (lin. ind.) A set of vectors $v_1, \dots, v_p \in V$ is <u>linearly independent</u> if $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$ only when $x_1 = \cdots = x_p = 0$.

Def. (basis) A <u>basis</u> of V is a subset of V that

(1) spans V and (2) is lin. ind.

Def. (finite dimensional) We say V is <u>finite dimensional</u> if V has a finite spanning set.

Facts: · If V is finite dimensional, then V has a basis that is finite. Moreover, when this is the case, every two bases of V must have <u>the same size</u>. That size is called the <u>dimension</u> of V.

Ex. Prove that $S_1 = \{1, t, t^2, t^3\}$ and $S_2 = \{2, t-1, 3t^2, t^3\}$

are both bases for P_3.

Pf: S_1: (1). Every general elt $v \in P_3$ is of the form $v = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$= a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3$, so $v \in \mathrm{Span}(S_1)$.

So S_1 spans V.

(2). If $x \cdot 1 + y t + z \cdot t^2 + w \cdot t^3 = \vec{0}$ for some scalars x, y, z, w.

Then since the zero poly $\vec{0}$ is $0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot t^3$, we must have $x = y = z = w = 0$.
(the zero vec in P_3)

So S_1 is lin. ind.

By (1) & (2), S_1 is a basis of P_3.

$S_2 = \{2, t-1, 3t^2, t^3\}$:

(1) A general elt $v \in P_3$ is of the form $v = a_0 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3$.

Rewriting, we get $v = \left(\frac{a_0}{2} + \frac{a_1}{2}\right) \cdot 2 + a_1 \cdot (t-1) + \frac{a_2}{3} \cdot (3t^2) + a_3 \cdot t^3$,

so $v \in \mathrm{Span}(S_2)$, so S_2 spans V.

(2) Suppose $x \cdot 2 + y \cdot (t-1) + z \cdot (3t^2) + w \cdot t^3 = 0$ for $x, y, z, w \in \mathbb{R}$.

Then $wt^3 + 3zt^2 + yt + (2x-y) = 0$, so $\begin{cases} w = 0 \\ 3z = 0 \\ y = 0 \\ 2x - y = 0 \end{cases}$, which implies $\begin{cases} w = 0 \\ z = 0 \\ y = 0 \\ x = 0 \end{cases}$

So $x = y = z = w = 0$, hence S_2 is lin. ind.

By (1) & (2), S_2 is a basis of P_3. $\square$

Thm. (unique decomposition) Let $B = \{v_1, \dots, v_p\}$ be a basis of V.

Then every vector $v \in V$ can be written as a <u>unique</u> linear comb.

of the form $v = c_1 \cdot v_1 + \dots + c_p v_p$ where $c_1, \dots, c_p \in \mathbb{R}$.

Pf: Same as before (for $\mathbb{R}^n$), see the proof worksheet.

Def. (Coordinate vectors) In the above setting, the vector $[v]_B := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^p$

is called <u>the coordinate vector of v with respect to B.</u>

Note: Taking the coordinate vector (w.r.t. a basis B) takes an abstract vec. v to a concrete column vector $[v]_B$ in $\mathbb{R}^p$. In fact, given a basis B of V, we can use the "coordinate mapping" $[\ ]_B : V \to \mathbb{R}^p$, $v \mapsto [v]_B$ to turn problems about V to problems about $\mathbb{R}^p$.

Eg $V = P_3$, $B = S_1 = \{1, t, t^2, t^3\}$, $p = 4$.

$$S_2 = \{\underset{v_1}{2}, \underset{v_2}{t-1}, \underset{v_3}{3t^2}, \underset{v_4}{t^3}\}.$$

Is $\{v_1, v_2, v_3, v_4\}$ a basis of V ?

$\Updownarrow$ fact: the two questions have the same answer.

Is $\left\{ [v_1]_B = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [v_2]_B = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [v_3]_B = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, [v_4]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^4$?

$\downarrow$

We can solve this quickly using EF.

More on coordinate mappings later ...

3. Linear maps.

Def: Let V, W be vector spaces. Then a map $T: V \to W$ is <u>linear</u>

if (1) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall v_1, v_2 \in V$

and (2) $T(cv) = cT(v) \quad \forall v \in V, \ c \in \mathbb{R}$.

Def. The kernel and image of a lin. map $T: V \to W$ are the set $\operatorname{Ker} T = \{v \in V : T(v) = 0\}$ and $\operatorname{Im} T = \{T(v) : v \in V\}$, respectively.

Prop: Let $T: V \to W$ be a linear map. Then

- $\operatorname{Im} T$ is a subspace of W.
- $\operatorname{Ker} T$ is a subspace of V.

Pf: Same as before, see worksheet.

Next time:

Coordinate mappings and their applications.