Math 2135. Lecture 28. Midterm II. Friday, Nov. 12. 10. 29. 2021.

Last time:

- det and area/volume
- def. and first examples of abstract vector spaces
  - for a nonempty set V to be a vector space, V need to have a binary operation $+: V \times V \to V$ and a scalar operation $\cdot : \mathbb{R} \times V \to V$ that satisfy eight properties/axioms
  - e.g. P_n : v.s of poly. w/ degree $\leq n$.

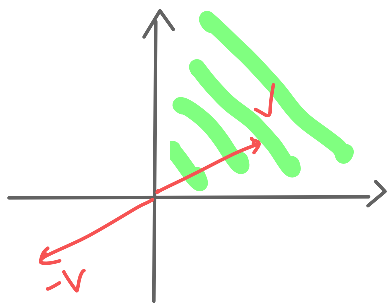
    function spaces.

Today:

- working with the axioms, basic proofs
- old notions/facts from $\mathbb{R}^n$ revisited.

1. Working with the def. of vector spaces.

Eg: Is the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0,\ y \geq 0 \right\} \subseteq \mathbb{R}^2$ a v.s. under the usual addition and scaling operations?



$+$: Is V closed under $+$? Yes.

$\cdot$: Is V ---- --- scaling (by arbitrary scalars)? No!

Since V is not closed under scaling, $\cancel{V}$ $(V, +, \cdot)$ is not a vector space.

Note: If we are to show a triple $(V, +, \cdot)$ is a vector space, we need to check V is closed under $+$ and $\cdot$ and satisfies the eight defining axioms.

E.g. Is $V = \{3 + a_1 t \mid a_1 \in \mathbb{R}\} \subseteq P_1$ a v.s. under the usual $+$ and $\cdot$?

$+$: $3 + t \in V$, $(3+t) + (3+t) = 6 + 2t \notin V$.

So V is not closed under addition, so V is not a v.s. $\square$

E.g. Is $V = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 0\}$ a v.s. under the usual $+$ and $\cdot$?

Soln: $+$: Take $f, g \in V$. Then $f(0) = 0$, $g(0) = 0$, so

$$(f+g)(0) \overset{\text{def.}}{=} f(0) + g(0) = 0 + 0 = 0.$$

So $f + g \in V$.

$\cdot$: EX: $\forall c \in \mathbb{R}$, $f \in V$, $c \cdot f \in V$.

· What about the eight axioms?

Sketch: (3) & (4): $\begin{cases} (3), & \text{the zero function does send } 0 \text{ to } 0, \text{ so it's in } V. \\ (4). & \text{if } f(0)=0 \text{ for a function } f: \mathbb{R} \to \mathbb{R}, \\ & \text{then } (-f)(0) = -\big(f(0)\big) = -0 = 0, \\ & \text{so Axiom (4) holds.} \end{cases}$

the other six axioms: they hold for V because they are arithmetic properties that hold for the larger set $W = \{ f: \mathbb{R} \to \mathbb{R} \}$.

In other words, V inherits these six properties from W.

Later, more precisely: V is a v.s. because W is a v.s. & V is a subspace of W.

E.g. Proving abstract properties of v.s. using the axioms.

Prop: In any abstract vector space V, we have $\underset{\mathbb{R}}{0} \cdot v = \underset{V}{\vec{0}}$ for all $v \in V$.

Note: We don't know what elts in V look like!

the zero vector, which should exist by Axiom 3.

Pf: We'll use the defining axioms. Let $v \in V$.

Since $0 + 0 = 0$ in $\mathbb{R}$, we have $(0+0) \cdot v = 0 \cdot v$

By Axiom (6) from Lecture 27, $(0+0) \cdot v \overset{(6)}{=} 0 \cdot v + 0 \cdot v$,

So $0 \cdot v + 0 \cdot v = 0 \cdot v$. Adding $-(0 \cdot v)$ to both sides,

we get $0 \cdot v + \boxed{0 \cdot v + \left(-(0 \cdot v)\right)} = 0 \cdot v + \left(-(0 \cdot v)\right)$, i.e., $0 \cdot v + \vec{0} = \vec{0}$,

(4) So $0 \cdot v = \vec{0}$ by Axiom (3). $\square$

2. Generalizing old notions/facts

Let V be a vector space.

(1). linear combinations.

Def: Let $v_1, v_2, \dots, v_p$ be vectors in V. Then a linear combination of $v_1, \dots, v_p$ is any vector of the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_p v_p \quad \text{where } a_1, \dots, a_p \in \mathbb{R}.$$

Eg: In P_3, $3t + 5t^2$ is a lin. comb. of t and t^2, as well as of $t + t^2$ and t^2

$$3t + 5t^2 = 3 \cdot (t + t^2) + 2 \cdot t^2.$$

(2). spans.

Def: The span of a set $\{v_1, \dots, v_p\}$ is the set of all lin. comb. of $v_1, \dots, v_p$.

eg: $3t + 5t^2 \in \text{Span}\{t + t^2, t^2\}$.

(3) Subspaces

Def. (As before) A subspace of V is a subset W of V s.t.

(1) $\underline{0_V} \in W$. (2) $\forall u, v \in W$, $u+v \in W$ (3) $\forall c \in \mathbb{R}$, $u \in W$, $c \cdot u \in W$.

the zero vec in V

Prop 1: (why these requirements?) Let W be a subset of V. Then W is vector space itself under the addition and scaling for V, i.e., $(W, +, \cdot)$ iff and only if W is a subspace of V in the sense of Def (*).

Prop 2: (spans are subspaces) Let S be any subset of V. Then the span of S is a subspace of V.

Pf: "Same as before" (details next time) **Next time:** more familiar notions / facts: proof practice.