Math 2135. Lecture 27.

10. 27. 2021.

Last time:

- new det. from old

- geometric significance of det: areas and volumes

2×2 det

Thm 1: $\begin{vmatrix} v_1 & v_2 \end{vmatrix}$ = Area (the ▱ with v_1 and v_2 as sides)

Thm 2: Given a lin. map $T: \mathbb{R}^2 \to \mathbb{R}^2$ and a ▱ P in $\mathbb{R}^2$,
$T(P)$ is another ▱ and

$$\text{Area}(T(P)) = \text{Area}\,P \cdot |\det A|$$

where A is the standard matrix of T.

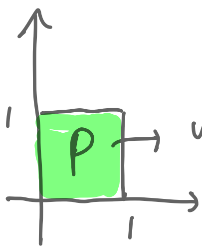
Today:

- det. and volume (3×3)

- start Ch4: abstract vector spaces

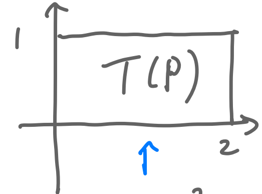
1. Det. and area/volume

Examples for Thm 2:



unit square

hor. stretch



$T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x \\ y \end{bmatrix}$, $A_T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

geometry: $\text{Area}(T(P)) = \text{base} \cdot \text{height}$

$= 2 \times 1 = 2$

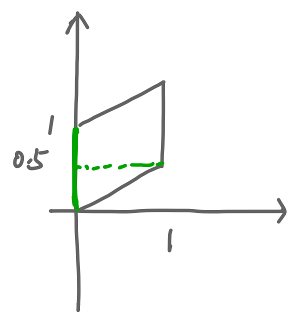
Thm 2: $\text{Area}(T(P)) = \text{Area}(P) \cdot \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$

$= (1 \times 1) \times 2 = 2$ ✓

ver. shear

$S: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y + x/2 \end{bmatrix}$, $A_S = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$



geometry: $\text{Area}(T(P)) = \text{base} \cdot \text{height}$

$= 1 \cdot 1 = 1$

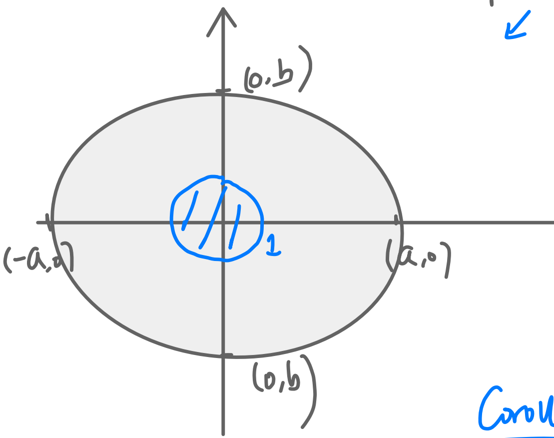
Thm 2: $\text{Area}(T(P)) = \text{Area}(P) \cdot \begin{vmatrix} 1 & 0 \\ 1/2 & 1 \end{vmatrix}$

$= (1 \times 1) \cdot 1 = 1.$ ✓

An application of Thm 2: Let $a, b > 0$. The points $(x, y) \in \mathbb{R}^2$ s.t.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

form an ellipse region R in $\mathbb{R}^2$.



Fact: We can obtain R from the unit disc $\mathcal{U}$ by stretching $\mathcal{U}$ hor. by a and ver. by b.

a linear transformation!

standard matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. $\left(\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax \\ by \end{bmatrix} \right)$

Corollary: $\text{Area}(R) = \text{Area}(\mathcal{U}) \cdot \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = (\pi \cdot 1^2) \cdot ab = \pi ab$.

We have thus derived the area formula for an ellipse.

When $a = b$, R is a disc and the formula recovers the area formula for discs.

Det. and volume:

Thm 1: (3.3.9.) Given a 3x3 matrix $A=[v_1|v_2|v_3]$, the volume of the parallelepiped determined by v_1, v_2, v_3 is $|\det A|$.

Thm 2. (3.3.10.) Given a parallelepiped P in $\mathbb{R}^3$ and a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$, then

can be generalize to other "nice" regions.

$$\mathrm{Vol}(T(P)) = |\det A_T| \, \mathrm{Vol}(P)$$

where A_T is the standard matrix of T.

2. Abstract vector spaces

Def: A <u>(real) vector space</u>, or a <u>vector space over $\mathbb{R}$</u>, is the data of a triple $(V, +, \cdot)$ where V is a nonempty set, $+$ is a map $+ : V \times V \to V$, $(v, w) \mapsto v+w$ called addition, and $\cdot$ is a map $\cdot : \mathbb{R} \times V \to V$, $(c, v) \mapsto c \cdot v$ called scalar multiplication which satisfy the following properties:

(1). $u+v = v+u \quad \forall u, v \in V$

(2). $(u+v)+w = u+(v+w) \quad \forall u, v, w \in V$.

(3). V has a special elt 0 called zero s.t. $u+0 = u = 0+u \quad \forall u \in V$.

(4). $\forall u \in V, \exists v \in V$ s.t. $u+v = 0$.

(5). $c(u+v) = cu + cv \quad \forall c \in \mathbb{R}, u, v \in V$.

(6). $(c+d) \cdot v = c \cdot v + d \cdot v \quad \forall c, d \in \mathbb{R}, v \in V$

(7). $(cd) \cdot v = c \cdot (d \cdot v) \quad \forall c, d \in \mathbb{R}, v \in V$

(8). $1 \cdot u = u \quad \forall u \in V$. We often refer to V as a vector space too. $\square$

New examples of vector spaces:

Apart from $\mathbb{R}^n$, there are many other types of vector spaces:

Eg. (polynomial space) Let $n \geq 0$. Let P_n be the set of all polynomials (with real coefficients) of degree at most n, i.e.,

$$P_n = \{ a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R} \}$$

Then P_n forms a vector space under the usual addition and scalar mult:

e.g. $n=4$, $(2t - t^2) + (1 - 2t + 3t^2 + 4t^3 + 5t^4) = 1 + 2t^2 + 4t^3 + 5t^4$. $\longrightarrow$ example of vec. addition

$3 \cdot (2t - t^2) = 6t - 3t^2$. $\longrightarrow$ example of scaling.

$(2t - t^2) + (-2t + t^2) = \underline{0}$ $\rightarrow$ the zero vector, $0 + 0 \cdot t + \cdots + 0 \cdot t^n$.

$\rightarrow$ an instance of Axiom (4).

Eg. (function spaces) The set V of all functions $\underline{\underline{f}} : \mathbb{R} \to \mathbb{R}$ forms a v.s. under function addition and scaling.

vector in V.

eg. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = \sin x$.

$g : \mathbb{R} \to \mathbb{R}$, $x \mapsto g(x) = 2x$

$f+g$: the function with $(f+g)(x) = f(x) + g(x) \quad \forall x \in \mathbb{R}$.

$\underset{\mathbb{R}}{\underset{\cap}{\underline{c}}} f$: . . . . . . $(cf)(x) = c \cdot f(x) \quad \forall x \in \mathbb{R}$.

The zero function: $0 : \mathbb{R} \to \mathbb{R}$ with $0(x) = 0 \quad \forall x \in \mathbb{R}$.

Ex: Can you define lin. ind. in an abstract v.s.?

Is {the sine function, the cosine function} lin. ind. in V?

Next time:

familiar notions/facts from $\mathbb{R}^n$, revisited.