Math 2135. Lecture 26.

10.25.2021.

Last time:

- Properties of determinants
  - effect of row operations, and related consequences
  - **Thm A:** for a square matrix A, A is inv. $\Leftrightarrow$ $\det A \neq 0$.
  - relationships between det and all things related to inv. (e.g. lin. ind.)

Today:

- More properties of det.

  Thm B: $\det(AB) = \det A \det B$ **Prop:** $\det(A^T) = \det A$.

- New determinants from old, with examples
- More applications of det.

1. Two more properties

Thm B. If A, B are $n \times n$ matrices, then $\det(AB) = \det A \cdot \det B$.

Pf: Skipped. See Section 3.2.

Corollary 1 (det & inverse) For any inv. square matrix A, $\det(A^{-1}) = \left(\det(A)\right)^{-1}$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{\det A}.$$

Pf: $\det A \cdot \det(A^{-1}) = \det(A \cdot A^{-1}) = \det(I) = \det\left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right) = 1$,

so $\det(A^{-1}) = 1/\det A$.

Corollary 2 (invertibility of products of matrices) Let $A_1, \dots, A_k$ be $n \times n$ matrices. Then $A_1 A_2 \cdots A_k$ is inv iff all of $A_1, \dots, A_k$ are invertible.

Pf: $A_1 A_2 \cdots A_k$ is inv $\overset{\text{Thm A}}{\iff} \det(A_1 \cdots A_k) \neq 0 \overset{\text{Thm B}}{\iff} \det(A_1)\det(A_2) \cdots \det(A_k) \neq 0$.

$\iff \det(A_1) \neq 0, \det(A_2) \neq 0, \cdots \det(A_k) \neq 0 \overset{\text{Thm A}}{\iff} A_1, A_2, \dots, A_k$ are all invertible.

Prop. For any square matrix A, $\det(A^T) = \det A$. ↘ pf in Section 3.2.

eg:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \quad \xrightarrow{\text{transpose}} \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad-bc.$$

Note: The properties of det. discussed in the last and this lecture allows us to compute det. of new matrices from det. of old matrices.

Eg.

- $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7 \quad \overset{\times 2}{\underset{\times 3}{\Longrightarrow}} \quad \begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = 7 \times 2 \times 3 = 42$$

- $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$.

  Find $\det AB$, $\det ABA^{-1}$, $\det A^3 B^T B^{-2}$.

Soln: Since $\det A = 3\times 1 - 6\cdot 0 = 3$, $\det B = 2\cdot 3 - 5\cdot 2 = -4$.

$\det(AB) = \det A \cdot \det B = 3\cdot(-4) = -12$,

$$\det(ABA^{-1}) = \det A \det B \det(A^{-1}) = \det A \det B \cdot \frac{1}{\det A} = \det B = -4.$$

$$\det(A^3 B^T B^{-2}) = \det(A^3) \det(B^T) \det((B^{-1})^2)$$

$$=$$

$$= (\det A)^3 \cdot \det B \cdot \left(\frac{1}{\det B}\right)^2$$

$$= (\det A)^3 \cdot \frac{1}{\det B}$$

$$= 3^3 \cdot \frac{1}{-4} = -\frac{27}{4}.$$

3. More applications of determinants

We have already used det. to detect invertibility and related notions.

In addition, det. can be used to ... (Let A be an $n \times n$ matrix.)

(1). directly finding matrix inverses.

Fact: If A is inv., then $$A^{-1} = \frac{1}{\det A}\left[\, x_{ij} \,\right]$$

where $x_{ij} = (-1)^{i+j} \det C_{ji}$ where C_{ji} is obtained from A by removing the jth row and ith col. of A for all i, j.

E.g. (2×2) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix},$$ recovering the familiar inv. formula for 2×2 matrices.

↓

if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is inv.

(2) directly solve linear systems / matrix equations

Consider the matrix eq. $Ax=b$ where A is inv and $b\in\mathbb{R}^n$.
$(n\times n)$

We expect a unique soln $x=A^{-1}b$.

Since A^{-1} can be computed directly via det, so can the soln.

Fact: If A is inv, then the unique soln $A^{-1}b$ of $Ax=b$ is given by the vector $x=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$ where $x_i=\dfrac{\det(A_i(b))}{\det A}$ where $A_i(b)=\left[v_1\middle|\cdots\middle|v_{i-1}\middle|b\middle|v_{i+1}\middle|\cdots v_n\right]$

if $A=\left[v_1\middle|v_2\middle|\cdots\middle|v_n\right]$

e.g. $\begin{bmatrix}3&-2\\-5&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$.

$$\Rightarrow\left\{\begin{aligned}x_1&=\frac{\begin{vmatrix}6&-2\\8&4\end{vmatrix}}{\begin{vmatrix}3&-2\\-5&4\end{vmatrix}}=\frac{40}{2}=20\\ x_2&=\frac{\begin{vmatrix}3&6\\-5&8\end{vmatrix}}{\begin{vmatrix}3&-2\\-5&4\end{vmatrix}}=\frac{54}{2}=27.\end{aligned}\right.$$

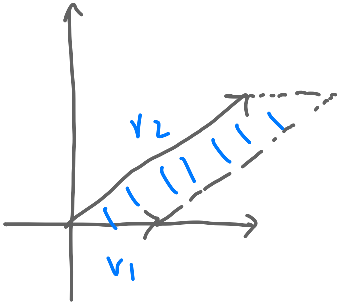
Check: $\left\{\begin{aligned}3\cdot 20-2\cdot 27&=6\\-5\cdot 20+4\cdot 27&=8.\end{aligned}\right.$

13) compute areas and volumes.
(2×2) (3×3)

Area: (a) Thm 1. (Thm 3.3.9) Given any 2×2 matrix $A = [v_1 | v_2]$, the area of the parallelogram determined (with sides) by v_1 and v_2 is given by $|\det A|$.

E.g.

$$\begin{array}{cc} A = & \left[\begin{array}{c|c} 2 & 4 \\ 0 & 3 \end{array}\right] \\ & \begin{array}{cc} v_1 & v_2 \end{array} \end{array}$$



$$\text{Area}(\square) = \text{base} \cdot \text{height}$$

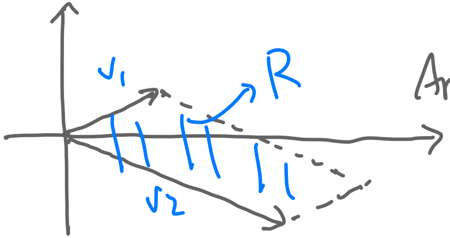
$\downarrow$
agree !
$\uparrow$

$$= 2 \cdot 3 = 6$$

$$|\det A| = |2 \cdot 3 - 4 \cdot 0| = |6| = 6$$

E.g.

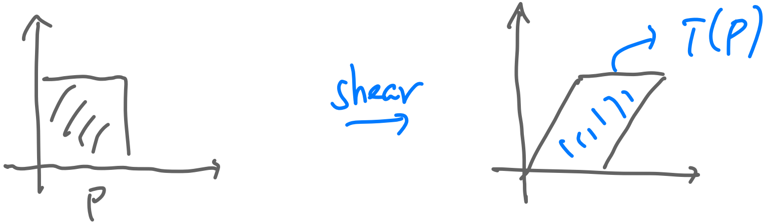
$$\begin{array}{cc} B = & \left[\begin{array}{c|c} 2 & 5 \\ 1 & -2 \end{array}\right] \\ & \begin{array}{cc} v_1 & v_2 \end{array} \end{array}$$



$$\text{Area}(R) \overset{\text{Thm1}}{=} |\det B| = 9$$

$$= |2 \cdot (-2) - 5 \cdot 1| = |-9|$$

(b) Thm 2 (Thm 3.3.10) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map w/ standard matrix A. If P is any parallelogram in $\mathbb{R}^2$, then the set $T(P)$ is also a ▱, and $\text{Area}(T(P)) = |\det A| \cdot \text{Area}(P)$.



Next time: finish Ch 3: more on Thm 2; det. and volume.

start Ch 4: abstract vector spaces.