Math 2135. Lecture 25.

10.22.2021.

Last time: · Determinants of square matrices (computation)

Notation: we'll often denote $\det[A_{ij}]$ by $|A_{ij}|$.

1×1: $\det[a] = a$; 2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$

larger: cofactor expansion.

Today: · Properties of determinants (with some proofs)

Point: to allow us to find new det from old quickly.

already mentioned two: Let A, B be $n\times n$ matrices.

Thm A. A is invertible $\Leftrightarrow$ $\det A \neq 0$.

Thm B. $\det(AB) = \det A \det B$.

1. Effect of elt. row operations on det.

Let A be an $n \times n$ matrix.

(1) effect of scaling

Fact 1. If we scale one row of A by a scalar c to obtain another matrix B, then

$$\det B = c \det A.$$

e.g.

$$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz \xrightarrow{\text{Scale Row 2}} \begin{vmatrix} x & y \\ cz & cw \end{vmatrix} = x \cdot cw - y \cdot cz$$

$$= c(xw - yz)$$

$$= c\begin{vmatrix} x & y \\ z & w \end{vmatrix}.$$

In general, Fact 1 can be proven using cofactor expansion along the scaled row.

Corollary 1. $\det(cA) = c^n \det A$ (if A is $n \times n$)

Pf: This follows from Fact 1 since cA can be obtained from A by scaling all the n rows by c.

E.g. $$\det(-A) = \det(-1 \cdot A) = (-1)^n \det A = \begin{cases} \det A & \text{if } n \text{ is even} \\ -\det A & \text{if } n \text{ is odd.} \end{cases}$$

(2) Effect of interchange

<u>Fact 2</u>. If we interchange two rows of A to obtain a matrix B,

then $\det B = -\det A$.

Eg.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{ei-fh} - b \underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{di-fg} + c \underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{dh-eg}$$

opposites of each other.

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = +a \underbrace{\begin{vmatrix} h & i \\ e & f \end{vmatrix}}_{hf-ei} - b \underbrace{\begin{vmatrix} g & i \\ d & f \end{vmatrix}}_{gf-id} + c \underbrace{\begin{vmatrix} g & h \\ d & e \end{vmatrix}}_{ge-hd}$$

Corollary 2. (a) If A has two identical rows, then $\det A = 0$.

(b) If one row of A is a multiple of another row, then $\det A = 0$.

E.g. $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 5 & 15 & 25 \end{vmatrix} \times 5 = 0$ since $R3 = 5R1$.

Pf: (1) Say $R_i = R_j$ in $\underset{A}{A}$. Then interchange R_i with R_j results in $\underset{\text{"}B\text{"}}{A}$.

So by Fact 2, $\det A = -\det A$. So $\det A = 0$.

(b). Ex. (useful example for hint: $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 5 & 15 & 25 \end{vmatrix} \overset{\text{Fact 1}}{=\!=} 5 \begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 1 & 3 & 5 \end{vmatrix} \overset{(a)}{=\!=} 5 \cdot 0 = 0.$)

13) Effect of replacements

Fact 3. If we add a multiple of a row in A to another row ($R_j \leftarrow R_j + cR_i$), to obtain a matrix B, then $\det A = \det B$.

E.g. $A: \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\text{"}B\text{"}: \begin{vmatrix} a & b \\ c-\lambda a & d-\lambda b \end{vmatrix} \overset{*}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ \lambda a & \lambda b \end{vmatrix} \overset{\text{Cor 2}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Ex: Check the claimed eq. (*).

<u>Corollary 3.</u> If two square matrices are related by a sequence of elt. row operations, then their determinants are either both 0 or both nonzero.

Pf: This follows immediately from Facts 1-3.

We can now derive Thm A: A is inv $\Longleftrightarrow$ $\det A \neq 0$.

Pf: A is inv $\overset{\text{the inv. thm}}{\Longleftrightarrow}$ $REF(A) = I_n$. $\Longleftrightarrow$ $\det\left(REF(A_n)\right) \neq 0$.

But $REF(A)$ is related to A by elt. row operations, so

$$\det(REF(A)) \neq 0 \Longleftrightarrow \det(A) \neq 0.$$

Thm A follows.

Note: Since we've related determinants to invertibility, we can expand the invertibility thm: for an $n\times n$ matrix A and its maps T $(x \mapsto Ax)$.

TFAE: A is inv; T is inv; the cols of A are lin. ind.;

· the cols of A span $\mathbb{R}^n$; · ··· ···

and · $\det A \neq 0$.

E.g. Is $\left\{ \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ lin. ind ?

$$\det \begin{bmatrix} 7 & 5 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 3 \end{bmatrix} = +7 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 7 \cdot (-3) = -21 \neq 0.$$

It follows from the extended invertibility thm that the given set is lin ind.

Next time: Thm B; $\det(A^T) = \det A$; more computations.