Math 2135. Lecture 24.

10. 20. 2021.

Last time:

- Coordinate vectors (to find them is solving vector equations)
- Rank/nullity of linear maps. Rank-Nullity Thm.

Done with Ch. 2.

Today. Start Ch. 3. Determinants of square matrices.

- To each square (n×n) matrix A we will associate a number $\det A$ called the determinant of A with remarkable properties.

Small n:

Def: The determinant of a 1×1 matrix $[a]$ is a.

. . . . . . 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$.

Recall: A 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc\neq 0$.

So: A is inv. iff $\det A\neq 0$.

In fact, determinants of square matrices have the following properties in general.

Thm 1. (invertibility via det.) Let A be an $n\times n$ matrix. Then A is invertible iff $\det A\neq 0$.

Thm 2. (det. is multiplicative) Let A, B be $n\times n$ matrices.

Then $\det(AB)=\det A \det B$.

Larger n:

But how can we compute det. of $n \times n$ matrices for general n?

Answer: Recursively, using so-called cofactor expansions.

Prop 1. (cofactor expansion along the first row). Let $A = [A_{ij}]$ be an $n \times n$ matrix.

Then $$\det A = \sum_{j=1}^{n} (-1)^{1+j} \cdot A_{1j} \cdot \det C_{1j},$$ where C_{1j} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the jth column.

Example:
$$A = \begin{array}{c} \begin{matrix} j=1 & j=2 & j=3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\det A = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix} + (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{1+3} \cdot 1 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= +1 \cdot 0 - 2 \cdot 1 + 1 \cdot 5 = 3$$

Prop 2 (cofactor expansion along arbitrary rows or columns):

Let $A = [A_{ij}]$ be an $n \times n$ matrix. Then for all $1 \le i, j \le n$, we have

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det C_{ij}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det C_{ij}$$

where C_{ij} is the matrix obtained from A by removing the ith row and jth column.

Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$

$$\det A \overset{\text{2nd row}}{=\!=} (-1)^{2+1} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} + * \cdot 0 \cdot * + * \cdot 0 \cdot * = -1 \cdot (-3) = 3$$

$$\det A \overset{\text{1st col}}{=\!=} (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix} + (-1)^{2+1} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} + * \cdot 0 \cdot * = 0 - 1 \cdot (-3) = 3.$$

Note: · (Once we've proved Thm 1), we can conclude that A is inv in the examples.

· As the examples show, it is convenient to use cofactor expansion along a row or column with a large number of zero entries.

Ex: Find the formula for $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Soln: We use cofactor expansion along the first row.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\cdot\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\cdot\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\cdot\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a\cdot(ei - hf) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + chd - ahf - bdi - ceg.$$

Ex. (Det. of triangular matrices)

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & a_{n-1\,n} \\ 0 & & & a_{nn} \end{bmatrix} = (-1)^{1+1} \cdot a_{11} \cdot \det C_{11}$$

SE(1,1)

→ SE of the (1,1) position

$$= +1 \cdot a_{11} \cdot \det C_{11}$$

$$= a_{11} \cdot a_{22} \cdot \det(\mathrm{SE}(2,2))$$

$$\vdots$$

$$= a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}.$$

eg.

$$\det \begin{bmatrix} 5 & \pi & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} = +5 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix} = 5 \cdot 1 \cdot \det \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = 5 \cdot 1 \cdot 4 \cdot \det [2]$$

$$= 5 \cdot 1 \cdot 4 \cdot 2 = 40.$$

Eg. Compute $\det A$ for $A = \begin{bmatrix} 3 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ -2 & -2 & 3 & -1 \end{bmatrix}$

$$\det A \overset{\text{2nd row}}{=\!=} (-1)^{1+2} \cdot 1 \cdot \det \begin{bmatrix} 1 & 3 & 0 \\ 5 & 0 & 1 \\ -2 & 3 & -1 \end{bmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -1 \end{bmatrix}$$

$$= -1 \cdot \left(1 \cdot \det \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} - 3 \cdot \det \begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} \right) + 2 \cdot \left((-1)^{2+3} \cdot 1 \cdot \det \begin{bmatrix} 3 & 3 \\ -2 & 3 \end{bmatrix} \right)$$

$$= -\left(-3 - 3 \cdot (-3) \right) + 2 \cdot (-15)$$

$$= -6 - 30 = -36.$$

Next time:

properties of det.