Math 2135. Lecture 23.

10. 18. 2021.

Last time: 1. Bases and unique decomposition

2. Bases and dimensions of $\underset{\text{"Im } T\text{"}}{\underline{\text{column}}}$, row and $\underset{\text{"Ker } T\text{"}}{\underline{\text{null}}}$ space of a matrix.

Today. · more example computations · the Rank-Nullity Thm.

1. Coordinate vectors

Setting: · We're given a basis (an ordered basis) $\mathcal{B} = (v_1, \cdots, v_p)$ of a subspace $V \subseteq \mathbb{R}^n$ and a vector $v \in V$.

· We want the unique vector $[v]_{\mathcal{B}} := \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ s.t. $v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$.

Def: (Coordinate Vector) The vector $[v]_B$ is called the coordinate vector of v with respect to B.

Recall: To find $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ is just to solve $x_1 v_1 + \cdots + x_p v_p = v$.

Example. Let $B = \left(v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right)$. Let $v = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \in \mathbb{R}^3$.

(1) Prove that B is a basis of $\mathbb{R}^3$.

Pf: There are 3 elts in B, and $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$ since

$$EF\left(\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$ has no zero rows.

It follows that B is a basis of $\mathbb{R}^3$.

(2) Find $[v]_B$. $\left(v = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix},\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}. \right)$

Soln: We need to solve $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 5 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \end{array}$$

$\Rightarrow x_1 = -2,\ x_2 = 2,\ x_3 = 1$, so $[v]_B = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. □

Note: Order matters. For example, if $B' = (v_2, v_1, v_3) = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right)$,

then $[v]_{B'} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

2. Bases of images and kernels of lin. maps

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. map.

Recall: For the standard matrix A of T, $\operatorname{im} T = \operatorname{Col} A$, $\ker T = \operatorname{Null} A$.

Example: Consider the linear map $T: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 3x \\ x+2y \\ 5x+4y \end{bmatrix}$.

Find a basis and the dimension of $\operatorname{Im} T$; repeat for $\operatorname{Ker} T$.

Soln: The standard matrix of T is $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 5 & 4 \end{bmatrix}$. ← the matrix from last lecture

We row-reduce A:

$$A \to \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & -6 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \end{bmatrix} \\ \ \ p \quad\ p \end{array}$$

It follows that $\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$ is a basis of $\operatorname{Im} T$, and $\dim(\operatorname{Im} T) = 2$.

For $\operatorname{Ker} T = \operatorname{Nul} A$ we need the reduced echelon form:

$$A \to \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}. \to \text{augmented matrix} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $\operatorname{Ker} T = \operatorname{Nul} A = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

So $\operatorname{Ker} T$ is the zero space.

Thus, the empty set $\emptyset$ is a basis of $\operatorname{Ker} T$ and $\dim(\operatorname{Ker} T) = 0$.

Fact: The space $\{\vec{0}\} \subseteq \mathbb{R}^n$ is always considered to have $\emptyset$ as a basis and 0 as dimension.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. (and let A be its standard matrix)

Def: · The <u>rank</u> of T is defined to be $\dim(\operatorname{Im} T)$.

this is just $\dim(\operatorname{Im} T) = \dim(\operatorname{Col} A) = \operatorname{rank}(A)$.

· The <u>nullity</u> of T is defined to be $\dim(\ker T)$.

$\dim(\ker T) = \dim(\operatorname{Null} A)$

Thm 1. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map (so its standard matrix A is $m \times n$).

Then $\operatorname{rank}(T) + \operatorname{nullity}(T) \overset{(*)}{=} n$ (which is $\dim$ (the domain) and also # cols of A).

Pf:

$$\left.\begin{array}{l} \cdot\ \operatorname{rank} T = \dim(\operatorname{Col} A) = \#\text{ pivot cols of } A \\ \cdot\ \operatorname{nullity} T = \dim(\operatorname{Null} A) = \#\text{ non pivot cols of } A \\ \cdot\ n = \#\text{ cols of } A \end{array}\right\} \Rightarrow (*).$$

Point: (1). $\mathrm{Rank}(T)$, $\mathrm{Nullity}(T)$ are numbers that encode surj / inj of T:

$T: \mathbb{R}^n \to \mathbb{R}^m$ (i) T is surj $\Leftrightarrow$ $\mathrm{Im}\,T = \mathbb{R}^m$ $\Leftrightarrow$ $\dim(\mathrm{Im}\,T) = \mathrm{Rank}(T) = m$. ("full-rank")

(ii) T is inj $\Leftrightarrow$ $\mathrm{Ker}\,T = \{0\}$ $\Leftrightarrow$ $\dim(\mathrm{Ker}\,T) = \mathrm{Nullity}(T) = 0$. ("trivial kernel")

(2). The Rank-Nullity Thm connects the notions of surj and inj.

e.g.: (deducing inj / surj from nullity / rank.)

(a) "square case": Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map.

By the above,

$$T \text{ is inj} \Leftrightarrow \mathrm{Nullity}\,T = 0 \overset{\text{Thm 1}}{\Longleftrightarrow} \mathrm{Rank}\,T = n \Leftrightarrow T \text{ is surj}.$$

We have thus recovered an equivalence from the invertibility thm.

(b). Let $T: \mathbb{R}^5 \to \mathbb{R}^3$. Then $\mathrm{Im}\,T \subseteq \mathbb{R}^3$, so $\mathrm{Rank}\,T \leq 3$. As $\mathrm{Rank}\,T + \mathrm{Nullity}\,T = 5$, $\mathrm{Nullity}\,T \geq 2$, so T cannot be inj. Now if $\mathrm{Rank}\,T = 1$, then $\mathrm{Nullity}\,T = 4$. Since $1 < 3$, T is not surj. Since $4 \neq 0$, T is not inj.

Next time: determinants.