Math 2135. Lecture 22.

10. 15. 2021.

Last time:

- Def of bases and dimension (common size of all bases) of a subspace V of $\mathbb{R}^n$. (spanning, lin. ind.)
- Bases of $\mathbb{R}^n$ (as a subspace of $\mathbb{R}^n$): · must have n elts;
  - a set $B \subseteq \mathbb{R}^n$ is a basis if two of (1) B spans $\mathbb{R}^n$ (2) B is lin. ind and (3) $|B| = n$ hold.
- Examples: $V = \mathbb{R}^3$ $\begin{cases} B = \{e_1, e_2, e_3\} \text{ is a basis;} \\ B' = \{e_1, e_2\} \text{ is not.} \end{cases}$

  $V = \text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} \subseteq \mathbb{R}^2$ $\begin{cases} B = \left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} \text{ is a basis of } V \\ B' = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\6\end{bmatrix}\right\} \text{ is not.} \end{cases}$

to be revisited

- Claims: A basis should be a maximally lin ind and minimally spanning set.

Today: · basis decomposition · basis / dim of row, cols and null spaces.

1. A unique decomposition property.

Thm. Let V be a subspace of $\mathbb{R}^n$ and let $B = \{v_1, \dots, v_p\}$ be a basis of V. Then for every $v \in V$, there are unique scalars $c_1, c_2, \dots, c_p$ s.t.

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p. \qquad (*)$$

Pf: · The scalars $c_1, c_2, \dots, c_p$ exist since B spans V.

· For uniqueness, suppose that $d_1, \dots d_p$ are scalars st $V = d_1 v_1 + \cdots + d_p v_p$. $(**)$

It suffices to show that $c_1 = d_1$, $c_2 = d_2$, $\dots$, $c_p = d_p$.

$(*)$ & $(**)$ $\Rightarrow$ $c_1 v_1 + \cdots + c_p v_p = d_1 v_1 + \cdots + d_p v_p$ $\Rightarrow$ $(c_1 - d_1) v_1 + \cdots + (c_p - d_p) v_p = 0$

$\xRightarrow{B \text{ is lin. ind.}}$ $c_1 - d_1 = c_2 - d_2 = \cdots = c_p - d_p = 0$ $\Rightarrow$ $c_1 = d_1, c_2 = d_2, \dots, c_p = d_p$. DONE. $\square$

Example. Consider $V = \mathbb{R}^2 \subseteq \mathbb{R}^2$ and let $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. $B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

Standard basis

(1) Prove that B_2 is a basis of V.

Pf: (i) $|B| = 2$ ✓, (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} \Rightarrow$ no zero row in EF, so B_2 span $\mathbb{R}^2$. (i) & (ii) $\Rightarrow B_2$ is a basis of $\mathbb{R}^2$.

(2) Let $v = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \in V$.

B_1: $\begin{bmatrix} 7 \\ 4 \end{bmatrix} = 7 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note: by the thm, this is the unique possible way to decomp. v into B_1.

B_2: $\begin{bmatrix} 7 \\ 4 \end{bmatrix} \overset{(\square)}{=} x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Note: by the thm, $(\square)$ has a unique soln. finding the decomp of v into B_2 is to solve the vector eq. $(\square)$

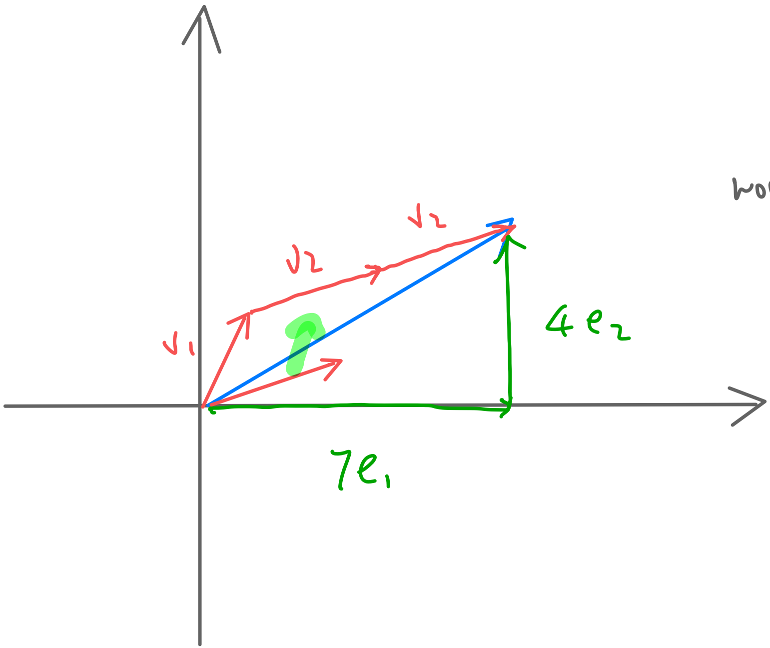
(ie, $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$)

EX: Solve $(\square)$.

Note: The "component vector" $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ will be important later.

In a picture:

$v = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, $\underbrace{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{B_2}$



would get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by the E.X.

2. Bases and dimension of Col(A), Row(A), Null(A) for a matrix A.

Thm. Let A be an arbitrary matrix and let $EF(A)$ be an echelon form of A.

(a) (Col(A)) The pivot columns of A (not $EF(A)$!) is a basis of $Col(A)$.

In particular, $\dim(Col(A)) = \#$ pivot cols of A = # pivots in $A/EF(A)$.

(b) (Row(A)) The pivot rows of $EF(A)$ (not A!) is a basis of $Row(A)$.

In particular, $\dim(Row(A)) = \#$ pivots in $A/EF(A)$

(c) (Null(A)) The constant vectors in the p.v.f. of the solns of $Ax=0$ is a basis of $Null(A)$. In particular,

$$\dim(Null(A)) = \text{\# ``free variables''} = \#\text{cols of } A - \#\text{pivots in } A.$$

Note: (a),(b) $\Rightarrow \dim(Row(A)) = \dim(Col(A))$; (a),(c) $\Rightarrow \dim(Col(A)) + \dim(Null(A)) = \#$ cols of A.

We'll postpone the proof the theorem.

Eg. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

EF

By the theorem,

- $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$ is a basis of $\mathrm{Col}(A)$. $\dim(\mathrm{Col}(A)) = 2$.

  Note: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} \right\}$ is not a basis of $\mathrm{Col}(A)$: any elt in their span has a zero last coordinate

- $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -8 \\ -12 \end{bmatrix} \right\}$ is a basis of $\mathrm{Row}(A)$, $\dim(\mathrm{Row}(A)) = 2$.

$\mathrm{Null}(A)$:

$(Ax=0)$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} p & p & f & f \end{array} \end{array}$$

$$\longrightarrow \begin{cases} x - z - 2w = 0 \\ y + 2z + 3w = 0 \end{cases} \longrightarrow \begin{cases} x = z + 2w \\ y = -2z - 3w \end{cases}$$

$$\mathrm{Null}(A) = \{ x \mid Ax = 0 \} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z+2w \\ -2z-3w \\ z \\ w \end{bmatrix} : z, w \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{z \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{p.v.f}} : z, w \in \mathbb{R} \right\}$$

So $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathrm{Null}(A)$; $\dim \mathrm{Null}(A) = 2$.

Eg.

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 5 & 4 \end{bmatrix}.$$

Ex: Find the bases and dimensions of $Col(A)$, $Row(A)$, $Nul(A)$.

Next time:

- Coordinate systems
- Rank, nullity, and the Rank-Nullity Thm.