Math 2135. Lecture 21.

10. 13. 2021.

Last time:

- examples of subspaces of $\mathbb{R}^n$ / $\mathbb{R}^m$.

(1) the extremes: $\{0\} \subseteq \mathbb{R}^n$; $\mathbb{R}^n \subseteq \mathbb{R}^n$.

(2) from matrices: any $m\times n$ matrix A $\to$ $\mathrm{Row}(A) \subseteq \mathbb{R}^n$, $\mathrm{Col}(A) \subseteq \mathbb{R}^m$, $\mathrm{Null}(A) \subseteq \mathbb{R}^n$.

(3) from linear maps: any lin. map $T: \mathbb{R}^n \to \mathbb{R}^m$ $\to$ $\mathrm{Ker}\,T \subseteq \mathbb{R}^n$, $\mathrm{Im}\,T \subseteq \mathbb{R}^m$.

Connecting (2) and (3): if A is the standard matrix of a linear map T, then $\mathrm{Null}(A) = \mathrm{Ker}\,T$, $\mathrm{Col}(A) \overset{*}{=} \mathrm{Im}\,T$.

$*$ $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ -y \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$.

$\mathrm{Col}(A) = \mathrm{Im}\,T = \left\{ x\begin{bmatrix} 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ More generally, $\mathrm{Col}(A) = \mathrm{Im}\,T = \mathrm{Span}\{T(e_1), \dots, T(e_n)\}$

Today: Bases and dimension of a subspace of $\mathbb{R}^n$.

1. Def of bases and dimension

Let $V \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$.

Def: (bases) A <u>basis</u> of V is a subset B of V that satisfies

(a) B spans V (b) B is linearly independent.

E.g. (1) $V = \text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} \subseteq \mathbb{R}^2$. $\Rightarrow B = \left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$ is a basis of V.

(a): def of V
(b). B has one, nonzero vect.

(1)' same V $\Rightarrow B' = \left\{\begin{bmatrix}2\\6\end{bmatrix}\right\}$ is a basis of V.

Ex: Explain this.

(2) $V = \mathbb{R}^2 \subseteq \mathbb{R}^2$ $\Rightarrow B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ is a basis.

Fact: Any two bases of V have the same size.

$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \to \begin{bmatrix}1 & 3\\0 & -2\end{bmatrix} \Rightarrow$ (a) (b) hold.

Def: (dimension) The <u>dimension</u> of V is the common size of all bases of V.

E.g. $\dim\left(\text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}\right) = 1$, $\dim \mathbb{R}^2 = 2$ by the earlier examples.

Ex. $\dim \mathbb{R}^n = n$.

2. Bases of $\mathbb{R}^n$ ("Extreme case" $V = \mathbb{R}^n \subseteq \mathbb{R}^n$ is the entire space)

Prop: (1) $\dim \mathbb{R}^n = n$. In particular, any basis of $\mathbb{R}^n$ must have exactly n elts.

(2) (Basis criterion) (Let $V = \mathbb{R}^n \subseteq \mathbb{R}^n$.) A set $B \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if any two of the following conditions hold: (i) B span $\mathbb{R}^n$; (ii) B is lin. ind; (iii) $|B| = n$.

Pf (sketch) (1) Because $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} = \{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

$B = \{v_1, v_2, \dots, v_k\} \longrightarrow M_B = [v_1 | \cdots | v_k]$

(2). (i) & (ii) $\Longrightarrow$ B is a basis of $\mathbb{R}^n$ by def.

(i) & (iii) $\Longrightarrow$ M_B is square (iii) and the cols of M_B span $\mathbb{R}^n$ $\xrightarrow{\text{inv. Thm.}}$ the cols of M_B are lin. ind $\Rightarrow$ (i) & (ii) ✓

(ii) & (iii) : similar.

E.g. Which of the following sets are bases of $\mathbb{R}^3$?

(1) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$: $\underline{|B| = 3}$ ✓. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow$ $\underline{B \text{ spans } \mathbb{R}^3}$ ✓. $\Rightarrow$ B is a basis of $\mathbb{R}^3$.

(2) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$: $|B| = 2 \neq 3$ $\Rightarrow$ B is not a basis of $\mathbb{R}^3$.

3. More examples and observations about bases/dimensions

Fact: If $V \subseteq \mathbb{R}^n$ is a subspace, then $\dim V \le n$. $\leftarrow$ to be justified later.

E.g. (1) $V = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

- Is $B = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ a basis of V? **Answer:** B is linearly ind., but B doesn't span V:

If B spans V, then $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \mathrm{Span}\, B$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ for some constant, which is not true.

(Or, write: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not equal to $c \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ for any $c \in \mathbb{R}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \notin \mathrm{Span}\, B$, so $\mathrm{Span}\, B \ne V$.)

(2) $V = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right\}$ $\quad B = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right\}$. Is B a basis of V?

What's $\dim V$? **Answer:** B spans V, but B is not lin. ind. since each elt in B is a mult. of the other. So B is not a basis of V.

Ex: $B' = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis of V, and $\dim V = 1$. (Key: $x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \end{bmatrix} = (x+2y) \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.)

Remark: Let V be a subspace of $\mathbb{R}^n$. Then

(*) A basis of V should be a minimal (most efficient) spanning set.

see previous example

(**) . . . . a maximal linear ind. set (can't be expanded).

Eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. How can we find a basis for

$\mathrm{Row}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$?

Next time:

- making (*), (**) precise.
- finding bases of various subspaces.