Math 2135. Lecture 20. Midterm solns are now on Canvas. 10.11.2021.

Last time:

- Invertible linear maps. Relationship with invertible matrices.
- Subspaces of $\mathbb{R}^n$: a subset S of $\mathbb{R}^n$ is a subspaces if
  (1) $0 \in S$ (2) $\forall u, v \in S$, $u+v \in S$ (3) $\forall v \in S$, $c \in \mathbb{R}$, $cv \in S$.

E.g.

- Zero space $\{0\} \subseteq \mathbb{R}^n$ is a subspace; $\mathbb{R}^n \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
- (Size constraint) If S is a subspace containing any nonzero vector v, then $cv \in S$ for all $c \in \mathbb{R}$, so S is infinite.
- A claim: For any set $\{v_1, \dots, v_k\}$ of vectors in $\mathbb{R}^n$, Span$\{v_1, \dots, v_k\}$ is a subspace of $\mathbb{R}^n$

Today.

- Pf of the claim.
- More examples of subspaces.

1. Spans are subspaces

Prop: Let $A = \{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$.

Then $\text{Span } A = \text{Span}\{v_1, \dots, v_k\}$ is a subspace of $\mathbb{R}^n$.

$$\text{Span}\{v_1, \dots, v_k\} = \{c_1v_1 + \dots + c_kv_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}.$$

Pf: Note that: (1) $0 = 0 \cdot v_1 + \dots + 0 \cdot v_k \in \text{Span } A$

(2) Let $u, v \in \text{Span } A$. So $u = c_1v_1 + \dots + c_kv_k$ and $v = d_1v_1 + \dots + d_kv_k$ for some $c_1, c_2, \dots, c_k, d_1, \dots, d_k \in \mathbb{R}$.

Thus, $u+v = (c_1v_1 + \dots + c_kv_k) + (d_1v_1 + \dots + d_kv_k) \overset{*}{=} (c_1+d_1)v_1 + \dots + (c_k+d_k)v_k$

So $u+v \in \text{Span } A$.

(3) Let $u \in \text{Span } A$, $c \in \mathbb{R}$. Then $u = c_1v_1 + \dots + c_kv_k$ for some $c_1, \dots, c_k \in \mathbb{R}$.

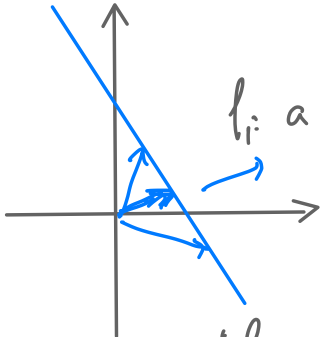
Thus, $cu = c(c_1v_1 + \dots + c_kv_k) \overset{*}{=} (cc_1)v_1 + (cc_2)v_2 + \dots + (cc_k)v_k$.

So $cu \in \text{Span } A$

By (1)–(3), Span A is a subspace of $\mathbb{R}^n$

Interlude: Some geometry.

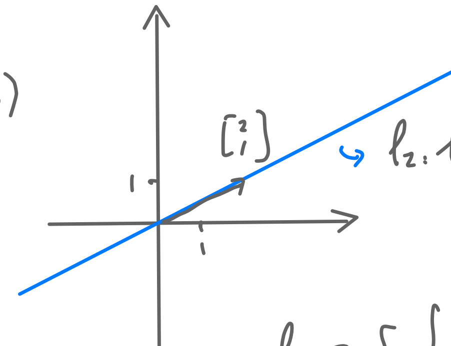
(a)



ℓ_1: a line not going through the origin

$0 \notin \ell_1$, so ℓ_1 is not a subspace.

(b)

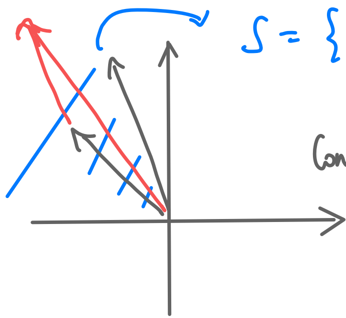


ℓ_2: line going through the origin.

$\ell_2 = \mathrm{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$,

so ℓ_2 is a subspace of $\mathbb{R}^2$.

(c)



$S = \left\{ \begin{bmatrix}x\\y\end{bmatrix} : \begin{matrix} x \le 0 \\ y \ge 0 \end{matrix} \right\}$

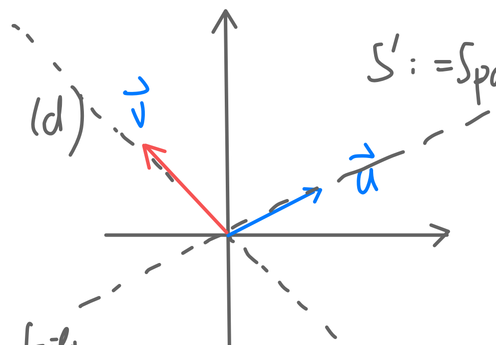
Condition (1) holds.

(2) ? Yes.

(3) ? Condition (3) fails.

So S is not a subspace of $\mathbb{R}^2$.

(d)



$S' := \mathrm{Span}\{\vec{u}, \vec{v}\}$ is a subspace of $\mathbb{R}^2$.

in fact, $S' = \mathbb{R}^2$.

2. Subspaces from matrices

Let A be an $m\times n$ matrix.

Def. (Row space / Column space) The row space of A is the span of the row vectors $\subseteq \mathbb{R}^n$; the column space of A is the span of the cols of A $\subseteq \mathbb{R}^m$.

e.g. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 4 \end{bmatrix} \longrightarrow \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \right\}$, $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

Note: Since they are spans, $\mathrm{Row}(A)$ is a subspace of $\mathbb{R}^n$ and $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Def. (Null space) The null space of A is the set $\mathrm{Null}(A) := \{ x \in \mathbb{R}^n \mid Ax = 0 \}$.

Prop: $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Pf: (1) Since $A\cdot\underline{0} = 0$, so $\underline{0} \in \mathrm{Null}(A)$. (2) Let $u, v \in \mathrm{Null}(A)$. Then $Au = 0$, $Av = 0$. So $A(u+v) \overset{*}{=} Au + Av = 0 + 0 = 0$, so $u+v \in \mathrm{Null}(A)$. (3) Let $u \in \mathrm{Null}(A)$, $c \in \mathbb{R}$. Then $Au = 0$. $A(cu) \overset{*}{=} cAu = c\cdot 0 = 0$, so $cu \in \mathrm{Null}(A)$.

✓✓✓ : DONE!

3. Subspaces from linear maps

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map

Prop: $\operatorname{Ker} T \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$; $\operatorname{Im} T \subseteq \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$.

First proof: We'll do the kernel part and leave the image part as an exercise.

(1) $T(0) = 0$ since T is linear, so $0 \in \operatorname{Ker} T$.

(2) Let $u, v \in \operatorname{Ker} T$. Then $T(u) = 0$, $T(v) = 0$.

$$T(u+v) \overset{*}{=} T(u) + T(v) = 0 + 0 = 0$$ where $*$ holds since T is lin.

So $u+v \in \operatorname{Ker} T$.

(3) Let $u \in \operatorname{Ker} T$, $c \in \mathbb{R}$. Then $T(u) = 0$.

$$T(cu) \overset{*}{=} cT(u) = c \cdot 0 = 0$$ where $*$ holds since T is linear.

So $cu \in \operatorname{ker} T$.

By (1) – (3), $\operatorname{ker} T$ is a subspace of $\mathbb{R}^n$.

Second proof: Let A be the standard matrix of T.

Then
$$\ker T = \{ x \in \mathbb{R}^n \mid T(x) = 0 \}$$
$$= \{ x \in \mathbb{R}^n \mid A \cdot x = 0 \}$$
$$= \mathrm{Null}(A),$$

so $\ker T$ is a subspace of $\mathbb{R}^n$ because $\mathrm{Null}(A)$ is.

Ex: $\mathrm{Im}\, T = \mathrm{Span}(\text{cols of } A)$ $\Leftarrow$ explain why

$$\Downarrow$$

$\mathrm{Im}\, T$ is a subspace of $\mathbb{R}^m$.

Next time: basis of subspaces of $\mathbb{R}^n$.