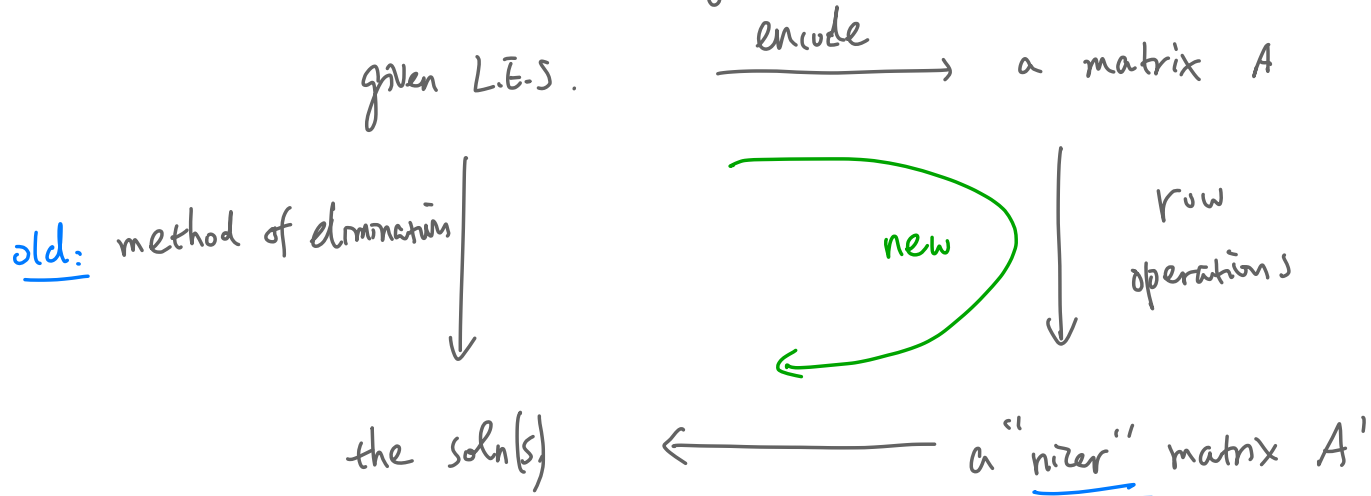


Last time: · Matrix encoding of systems of linear equations (L.E.S.)

· New scheme for solving L.E.S.



· The row ops. are motivated by the method of elimination. They are ..... (next page)

encodes an equiv. easier to get the solns from the matrix

# 1. Elementary row operations.

Def. By an elementary row operation on a matrix  $A$ , we mean one of the following three kinds of operations:

(E1) (Interchange) Swap two rows.  $R_i \leftrightarrow R_j$

(E2) (Scaling) Multiply (every entry in) a row by a nonzero constant  $c$ .  
 $R_i \rightarrow cR_i$

(E3) (Replacement) Add a multiple of a row  $R_j$  to another row  $R_i$  to replace  $R_i$ .  
 $R_i \leftarrow R_i + c \cdot R_j$

Prop. (Elementary row ops preserve equivalence of L.T.S.'s)

If a seq. of elt row op. take a matrix  $A$  to another  $A'$ , then  $A$  and  $A'$  encode equivalent L.T.S.'s.

## Unanswered questions:

1. What counts a "nizer" matrix  $A'$  from Page 1?

A: "Echelon form"  $\rightarrow$  matrices in echelon form make solns easy to find.

2. Can we always get to the nizer encoding matrix via only the three elt. row operations? If so, how?

A: Yes, we can always transform a matrix  $A$  to an echelon form by an algorithm called Gaussian elimination.

## 2. Another example (from the last lecture)

$$(1). \begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{E1} \begin{bmatrix} \textcircled{2} & 0 & -1 & 5 \\ 0 & \textcircled{2} & 1 & 1 \\ 0 & 0 & \textcircled{3} & 9 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E3} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R2 \leftarrow R2 - R3$

↓  
this turns out to be in 'echelon form'.

$$\xrightarrow{E3} \begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{cases} x = 4 \\ y = -1 \\ z = 3 \end{cases}$$

$R \leftarrow R_1 + R_3$



### 3. Echelon forms.

Def. A matrix  $T$  is in echelon form if it satisfies:

(1) All nonzero rows appear above any zero row.  
(all entries are zero)

(2) Every leading entry in a row is in a column strictly to the right of the column of the leading entry in the above row.  
(= first nonzero entry)

(3) All entries in a column below a row leading entry are zeros.

(In fact, (2) and (3) are equivalent.)

If a matrix  $A$  is in echelon form, then we say  $A$  is in reduced echelon form if it further satisfies the conditions.

(a). Every row-leading entry is 1.

(b). Every entry above a row-leading entry in a column is zero. (So that row-leading entry is the only nonzero entry in the column, because entries below it are zero by (3).)

2. Another example (from the last lecture)

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$$(1). \begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{E1} \begin{bmatrix} \textcircled{2} & 0 & -1 & 5 \\ 0 & \textcircled{2} & 1 & 1 \\ 0 & 0 & \textcircled{3} & 9 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E3} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R2 \leftarrow R2 - R3$

✓ echelon form.  
not reduced  
echelon.  
this turns out to be in 'echelon form'.

$$\xrightarrow{E3} \begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \leftarrow \text{reduced echelon form}$$
$$\begin{cases} x = 4 \\ y = -1 \\ z = 3 \end{cases}$$

$R \leftarrow R_1 + R_3$

# Examples / Non-examples.

Below we use  $\square$  to stand for nonzero

numbers and  $*$  for arbitrary numbers.

(1). 
$$\begin{bmatrix} 0 & 0 & 0 \\ \square & * & * \end{bmatrix}$$

not echelon, zero row above nonzero row.

$\downarrow$  E1

(2). 
$$\begin{bmatrix} \square & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

echelon, not reduced echelon unless  $\square = 1$

(3). 
$$\begin{bmatrix} \square & * & * & \pi & 0 & 3 \\ 0 & \square & * & * & 0 & 5 \\ 0 & 0 & 0 & 0 & \square & 0 \end{bmatrix}$$

echelon, reduced echelon if and only if all the three  $\square$ 's equal 1 and  $* = 0$ .

## Theorem.

(1) Every matrix can be transformed to a matrix in echelon form.

Moreover, we can do so using only the three elementary row operations

(2) Every matrix can be transformed to a unique  $\left( \begin{array}{l} \text{one and} \\ \text{only one} \end{array} \right)$  matrix in reduced echelon form.

Reason: Property (1) can be achieved using  $E1$ ,  
... (2) / (3) ...  $E3$ ,  
(a) ...  $E2$ ,  
(b) ...  $E3$ .

$\left( \begin{array}{l} \text{the uniqueness of the} \\ \text{reduced echelon form} \\ \text{is harder.} \end{array} \right)$

Ex. Solve  $\begin{cases} x+y+z = 6 \\ x+2y+z = 8 \\ 2y-z = -2 \end{cases}$  by transforming its encoding matrix

into its reduced echelon form using elt. row ops.

Next time: Row reduction algorithm / Gaussian elimination  
to obtain (reduced) Echelon forms.