# Math 2135. Lecture 19.

10.08.2021.

**Last time:** · Algorithm for testing invertibility and finding inverses for square matrices

$A \rightsquigarrow [A \mid I] \longrightarrow [EF(A) \mid *]$

→ $EF(A)$ is not regular $\Rightarrow$ $A$ is not inv.

→ $EF(A)$ is regular, go on

$[REF(A_n) \mid C] = [I_n \mid C]$

$C = A^{-1}$.

· The Invertibility Thm: for a square matrix $\underset{n\times n}{A}$ and its associated map $T: x \mapsto Ax$.

TFAE: (a) $A$ is inv; (b) The cols of $A$ are lin.

(c) The cols of $A$ span $\mathbb{R}^n$; (d) ... (i). 9 conditions.

**Today:** · Invertible linear maps · Subspaces of $\mathbb{R}^n$.

## 1. Invertible linear maps.

**Ex:** If $T$ is invertible, its inverse is unique.

**Def:** A linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> if there is another map $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $S \circ T = Id$ and $T \circ S = Id$ where $Id$ denotes the identity map $(x \mapsto x)$ on $\mathbb{R}^n$. When such a map $S$ exists, we call $S$ an inverse of $T$.

**Note:** · Under the bijections

$$\left\{\begin{array}{c}\text{linear maps} \\ \text{from } \mathbb{R}^n \text{ to } \mathbb{R}^m\end{array}\right\} \rightleftarrows \{m \times n \text{ matrices}\}.$$

$T \longmapsto M_T$, the standard matrix

$T_A \longleftarrow A$
$(x \mapsto Ax)$

(i) $M_{T_1 \circ T_2} = M_{T_1} \cdot M_{T_2}$ as matrices

and (ii) $T_{AB} = T_A \circ T_B$ as maps.

· $T = Id \iff M_T = I$, the identity matrix.

**Thm:** Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix $A$.

Then $T$ is invertible if and only if $A$ is invertible.

Moreover, if $T$ is invertible, then its inverse has standard matrix $A^{-1}$.

**Corollary:** In the above setting ($A: n\times n$, $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = A\cdot x \ \forall x \in \mathbb{R}^n$), we have $T$ is invertible iff $A$ is invertible, iff any of the other eight conditions (e.g. the cols of $A$ span $\mathbb{R}^n$) holds.

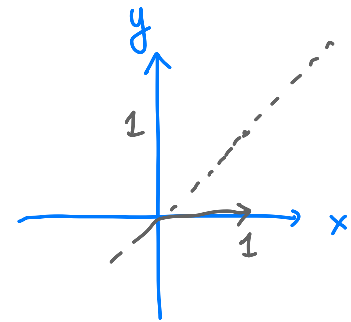
**Pf sketch:**

- If $T$ is invertible, we can use (i) to show that $M_{T^{-1}} = A^{-1}$.

- $A$ is invertible, we can use (ii) to show that $T_{A^{-1}} = T^{-1}$.

**Ex:** Fill out the details.

**E.g.** Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map that first reflects points across $y = x$ and then reflects points across the $x$-axis. Is $T$ invertible? Is so, find the formula for the inverse map $T^{-1}$.




**Strategy:**

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d, "\text{standard matrix}"'] & T^{-1}\,? \\
A \arrow[r, "\text{inv. alg.}"] & A^{-1} \arrow[u]
\end{tikzcd}$$

**Soln:** The standard matrix $A$ of $T$ is $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$\left[A \mid I_2\right] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$I_2$

So $A$ is inv and $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

By the thm, $T$ is inv and $T^{-1}$ is given by $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. □

**E.g.** Find the inverse map of the linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x-3y \\ y \end{bmatrix}$.

**Soln:** The standard matrix $A$ of $T$ is $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right] \checkmark$$

$I_2$

So $A$ is inv and $A^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

It follows that $T$ is invertible and $T^{-1}$ is given by

$$\underbrace{T^{-1}}_{S}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ y \end{bmatrix} \quad \left( S: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+3y \\ y \end{bmatrix} \right)$$

**Check:**

$$\begin{cases} S \circ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} x-3y \\ y \end{bmatrix}\right) = \begin{bmatrix} (x-3y)+3y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \\ T \circ S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} x+3y \\ y \end{bmatrix}\right) = \begin{bmatrix} (x+3y)-3y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{cases}$$

$\square$

## 2. Subspaces of $\mathbb{R}^n$

The set $\mathbb{R}^n$ is an example of "vector spaces".

We'll define subspaces of $\mathbb{R}^n$.

**Def:** A nonempty set $S \subseteq \mathbb{R}^n$ of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

(1) $\vec{0} \in S$.

(2) (closure under addition) $\forall u, v \in S$, $u+v \in S$.

(3) (closure under scaling) $\forall u \in S$, $c \in \mathbb{R}$, $cu \in S$.

**Note:** · Condition (3) implies Condition 1: if $\forall u \in S$, $c \in \mathbb{R}$, $cu \in S$, then when $c = 0$. $c \cdot u = 0 \cdot u = \vec{0} \in S$ for any $u \in S$, so (1) holds.

**Examples / Non-examples:**

(a) $n=2$, $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$ : not a subspace, Property (1) already fails. in fact, (2) and (3) fail as well.

(b) $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

Property (1) holds, but (2) and (3) fail, so $S$ is not a subspace.

(c). $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$ is a subspace of $\mathbb{R}^n$ because Properties (1)-(3) all hold.

(d). $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. is not a subspace of $\mathbb{R}^3$ : (2) and (3) fail.

(e). If $S \subseteq \mathbb{R}^n$ contains any non zero elt, then $S$ cannot be a subspace of $\mathbb{R}^n$ unless it is an infinite set.

(f). Let $n$ be any positive integer and let $v$ be any elt in $\mathbb{R}^n$.

Then $\mathrm{Span}\{v\} = \{ cv : c \in \mathbb{R} \}$ is always a subspace of $\mathbb{R}^n$.

(f') In fact, given any set $\{v_1, \ldots, v_k\}$ of elts in $\mathbb{R}^n$.

the span $\mathrm{Span}(\{v_1, \ldots, v_k\})$ is always a subspace of $\mathbb{R}^n$.

Next time: . prove f) and (f').

. more on subspaces.