Math 2135. Lecture 18.

10.06.2021.

Last time: · properties of inversion: if A, B are inv. $n\times n$ matrices, then

$$(A^{-1})^{-1} = A \ ; \quad (AB)^{-1} = B^{-1}A^{-1} \ ; \quad (A^T)^{-1} = (A^{-1})^T .$$

· EF criterion for inv: a square matrix A is inv $\iff$ $REF(A) = I_n$ $\iff$ ($EF(A)$ is "regular")

Preparation for proof: every elementary row op can be achieved by mult. w/ an inv. matrix.

· Inversion algorithm: given an $n\times n$ matrix A, compute $EF([A \mid I_n])$. If $EF(A)$ is not regular, then A is not inv. If $EF(A)$ is regular, then the right half of $REF([A \mid I])$ is A^{-1}.

Today: · example applications of the inv. algorithm · the invertibility thm.

1. Inversion examples

(a) Determine if the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ is invertible. Find A^{-1} if so.

Soln: $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right]}_{EF(A)} \longrightarrow \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4/5 & -3/5 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right]$. The above row operations imply that $A^{-1} = C$.

$REF(A) = I_2$ $\quad C$

(b) Solve the equations $Ax = \underbrace{\begin{bmatrix} 5 \\ 10 \end{bmatrix}}_{b_1}$ and $Ax = \underbrace{\begin{bmatrix} 0 \\ -7 \end{bmatrix}}_{b_2}$.

Sketch: $x = A^{-1}b = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $\quad x = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 21/5 \\ -14/5 \end{bmatrix}$.

(c). Determine if the matrix $A = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$ is invertible; find A^{-1} if so.

Soln:

$$\left[\begin{array}{ccc|ccc} 0 & 3 & -5 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & -9 & 15 & 0 & 4 & 1 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{array}\right]$$

EF(A) is not regular, so A is not invertible.

Let's make this official: a square echelon form is regular if

$$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \ddots \end{bmatrix}$$

it has a pivot in every row and column.

2. The Invertibility Thm.

Thm: Let A be $n\times n$ matrix and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding map given by $T(x) = A\cdot x \ \forall x \in \mathbb{R}^n$. Then TFAE.

square!

(a) A is invertible.

(b) The cols of A are lin. ind.

(c) The cols of A span $\mathbb{R}^n$.

(d) All cols of A are pivot.

(e) All rows of A are pivot.

(f) The map T is inj.

(g) The map T is surj.

(h) $\ker T_A = \{0\}$.

(i) $\operatorname{Im} T_A = \mathbb{R}^n$.

Remarks:

- Recall that $\underbrace{\text{(b)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(f)} \Leftrightarrow \text{(h)}}_{L}$ and $\underbrace{\text{(c)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(g)} \Leftrightarrow \text{(i)}}_{R}$ for any matrix and its associated map.
- Since A is square, every condition in L is equiv. to every condition in R. ((d) $\Leftrightarrow$ # pivot cols $= n \Leftrightarrow$ # pivots $= n \Leftrightarrow$ # pivot rows $= n \Leftrightarrow$ (e))

  Thus, conditions (b) – (i) are pairwise equivalent.
- We saw that (a) $\Rightarrow$ (b): if A is invertible, then the eq. $Ax = 0$ has a unique soln $x = A^{-1}0 = 0$, the trivial sol $\Rightarrow$ the cols of A are lin. ind.

- To prove the theorem, it remains to show that (a) follows from any of (b) – (i), i.e., that A is inv. if any of (b) – (i) holds.

Pf: As remarked, it suffices to show that A is inv. if (d) holds.

(d) every col is pivot $\Rightarrow$ EF(A) is regular, $\mathrm{REF}(A) = I_n$

$\Rightarrow$ there are row op. $R_1, R_2, \dots, R_k$ s.t. $R_1 \circ R_2 \cdots \circ R_k(A) = I_n$

$\Rightarrow$ there are invertible matrices $M_1, \dots, M_k$ s.t. $\underbrace{M_1 \cdot M_2 \cdot \cdots \cdot M_k}_{M} \cdot A = I_n$.

$\Rightarrow$ $M \cdot A = I_n$ where $M = M_1 \cdots M_k$ is inv. $\Rightarrow$ $A = M^{-1}$ $\Rightarrow$ A is inv, with $A^{-1} = M$. $\square$

Note: $A^{-1} = M = M_1 \cdots M_k = M_1 \cdots M_k \cdot I_n = R_1 \circ R_2 \circ \cdots \circ R_k(I_n)$,

that is, the row op. taking A to I_n takes I_n to A^{-1}.

which is why the inversion algorithm works when A is inv.

Next time, invertible lin. maps.

E.g. Recall $\mathrm{EF}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right)$ has a nonpivot row, so A is not invertible.