Math 2135. Lecture 17.

10.04.2021.

Last time:

- Invertibility allows cancellation.
- 2x2 inverses: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$, in which case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- transposes, powers, and their properties.

  most interesting: $(AB)^T = B^T A^T$

Today:

- more properties of inversion
- echelon form criterion for invertibility
- algorithm for testing inveAtibility and finding inverses.

1. More properties of inversion.

Proposition: Let A, B be invertible $n \times n$ matrices. Then

(1) (Inversion is involutive) $(\underbrace{A^{-1}}_{A'})^{-1} = A$

$\left(\begin{array}{l} A' \text{ is the inverse of } A. \\ \Rightarrow A'A = AA' = I_n \\ \Rightarrow A = (A')^{-1}. \end{array} \right)$

(2) ("Sock-shoe principle")

AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

(applies to more factors as well; e.g. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$)

(3) (Inversion commutes with transposition)

A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$

Ex: Given an nonzero scalar $c \in \mathbb{R}$, is cA necessarily invertible? If so, what's the inv?

$\boxed{\text{Yes. } (cA)^{-1} = c^{-1}A^{-1}}$

Examples:

(2) $\overset{\text{LHS}}{(AB)^{-1}} = \overset{\text{RHS}}{B^{-1}A^{-1}}$. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$,

$\rightarrow \begin{cases} AB = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \xrightarrow[\text{LHS}]{\text{directly find } (AB)^{-1}} (AB)^{-1} = \frac{1}{6-4}\begin{bmatrix} 6 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1/2 \\ -2 & 1/2 \end{bmatrix} \\ A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \xrightarrow[\text{RHS}]{B^{-1}\cdot A^{-1}} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & -1/2 \\ -2 & 1/2 \end{bmatrix} \end{cases}$

(3) $\underbrace{(A^T)^{-1}}_{X} = \underbrace{(A^{-1})^T}_{Y}$ $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$\rightarrow \begin{cases} A^T = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \xrightarrow{\text{LHS}} (A^T)^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \\ A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \xrightarrow{\text{RHS}} (A^{-1})^T = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \end{cases}$ prop

Pfs: We'll do (2) together, (1), (3), Ex have similar proofs.

(Idea: to prove a matrix X is the inverse of a matrix Y, show

$XY = YX = I$.)

$X = AB$

$Y = B^{-1}A^{-1}$

① $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

② $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

① & ② $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$. $\square$

2. Echelon form characterization of invertibility.

Thm 1. Let A be an $n\times n$ matrix. Then TFAE.

(1) A is invertible.

(2) Every EF has the "regular staircase shape"

(3) $REF(A) = I_n$.

e.g. nonsquare $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$, not "regular"

Square $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, regular.

$\begin{bmatrix} * & & & \\ & * & & \\ 0 & & * & \\ & & & * \\ & & & & * \end{bmatrix}$ with a pivot in every row and col.

E.g. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xrightarrow{(2)}$ A is inv.

$B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{(2)}$ B is not invertible.

not the case since $EF(B)$ has a non-pivot col.

Note: The conclusions are to be expected:

- For A, $1\times 4 - 2\times 3 = -2 \neq 0$, so A should be inv. by earlier results.
- If B were invertible, $Bx = 0 \Rightarrow x = B^{-1}0 = 0$, so the cols of B would be lin. ind.

We will not prove the theorem yet.

But we'll note a key fact: each elementary row operation R can be realized by matrix multiplication with an invertible matrix on the left: $R(M) = \underbrace{E_R}_{\text{invertible}} \cdot M$

e.g. $M = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

· Scaling. 2 row, scale by 3 $\quad R\left(\quad M \quad \right) = \begin{bmatrix} 2 & 1 & 0 \\ 9 & 6 & 3 \\ 0 & 0 & 5 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot M$

· interchange. Row 1 $\longleftrightarrow$ Row 3 $\quad R(M) = \begin{bmatrix} 0 & 0 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix},$

· replacement. $R2 \leftarrow R2 - 2R1 \quad R(M) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

<u>Note</u>: The underlined matrices are indeed invertible.

3. The inversion algorithm

Let A be an $n\times n$ matrix. e.g. $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} =: A$

To determine if A is inv. and find A^{-1} if so, we can use the following algorithm.

Step 1: Form the matrix $B = \left[\begin{array}{c|c} A & I_n \end{array}\right]$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \rightarrow B = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right]$

Step 2: Row-reduce B to red. Echelon form. $B \rightarrow \left[\begin{array}{c|c} \overset{\mathrm{REF}(A)}{C} & D \end{array}\right]$

e.g. $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -6 & -5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 5/6 & 1/6 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2/3 & -1/3 \\ 0 & 1 & 5/6 & 1/6 \end{array}\right]$

Step 3: If $C = \mathrm{REF}(A) = I_n$, then A is inv and $D = A^{-1}$. e.g. $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2/3 & -1/3 \\ 5/6 & 1/6 \end{bmatrix}$

If $C \neq I_n$, then A is not inv.

Next time: examples; justifying Thm 1 and the algorithm.