Math 2135. Lecture 16.

09.28.2021.

Last time: (1). Under the bijection $\{\text{linear maps}\}$ and $\{\text{matrices}\}$,

$(T: \mathbb{R}^n \to \mathbb{R}^m) \longmapsto M_T$: $m\times n$ standard mat. of T

$T_A: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ $\longleftarrow$ A : $m\times n$

matrix mult. corresponds to composition of linear maps:

$$M_{(T_A\circ T_B)} = A\cdot B \text{ ; or } M_{S\circ T} = M_S \cdot M_T .$$

(2). (1) implies that matrix multiplication is associative. $((AB)C = A(BC))$

(3). A square matrix A is invertible if there is a matrix B s.t. $AB = BA = I_n$. $\to$ B is ~~an~~ "the" inverse of A.

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(4). Invertibility allows cancellation:

$$AB = AC, \quad A \text{ inv} \implies (A^{-1})AB = (A^{-1})AC \implies B = C.$$

$$Ax = b, \quad A \text{ inv} \implies (A^{-1})Ax = A^{-1}b \implies x = A^{-1}b.$$

Prop: If A is a square, $n \times n$ matrix with an inverse, then the inverse is unique. i.e., if B, C are matrices s.t.

$$AB = BA \overset{*}{=} I_n, \qquad AC = \underbrace{CA = I_n}_{\star}.$$

then $B = C$.

Pf: $$C = I_n C \overset{*}{=} (BA)C = BAC = B(AC) \overset{\star}{=} BI_n = B. \quad \square$$

Today:

- 2×2 inverses.
- powers and transposes of matrices.

1. 2×2 inverses.

Warm-up: A 1×1 matrix is just $A=[a]$. To have an inverse is to have another matrix $B=[b]$ s.t. $AB=[ab]$ and $BA=[ba]$ is 1.

Thus, A is inv iff $a\neq 0$. If $a\neq 0$, $A^{-1}=[1/a]$.

Thm. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

(1) If $ad-bc=0$, then A is not invertible. (e.g. $\begin{bmatrix} 1 & 2 \\ 7 & 14 \end{bmatrix}$)

(2) If $ad-bc\neq 0$, then A is invertible and has inverse $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Pf: (1) " $ad-bc=0 \Rightarrow A$ is not invert. " Ex. $3/4 \cdot 3 + (-1/4) 4$

(2). " $ad-bc \neq 0 \Rightarrow A$ is inv and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. "

Note: To check a matrix B is the inv. of another matrix C,

check $BC = CB = I_n$.

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & db-bd \\ -ca+ac & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{1}{ad-bc}\right) \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

E.g. Determine if $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ is inv. and solve $Ax = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Soln: $2 \cdot 3 - 1 \cdot 2 = 4 \neq 0$, so A is inv, and $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 \\ -3/4 & \frac{1}{2} \end{bmatrix}$.

$$Ax = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow x = A^{-1}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 \\ -3/4 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/4 \\ -1/4 \end{bmatrix}.$$

□

2. Powers and transposes.

- Powers : If a matrix A is square, say $n \times n$, then we can take the powers $A^k = A \cdot \dots \cdot A$ (k copies) of A for any $k \geq 1$.
- Transposes : Given an $m \times n$ matrix A, the $n \times m$ matrix B where $B_{ij} = A_{ji}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ is called the transpose of and is denoted by A^T. (e.g. $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{bmatrix} \to A^T = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}$)

Note: Transposition switches rows and cols.

Recall that we are interested in how matrix operations interact. Here are some nice properties :

Prop: (a) transpose & mult : $\underset{m\times n}{(A}\,\underset{n\times p}{B)}^T = \underset{p\times n}{B^T}\,\underset{n\times m}{A^T}$ e.g: $(A^T)^T = A$.

(b) transpose & inversion : A inv. $\Rightarrow$ A^T is inv, and $(A^T)^{-1} = (A^{-1})^T$.

(c) $(A+B)^T = A^T + B^T$

(d) $(rA)^T = r \cdot A^T$.

E.g. (a). Take $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Check $(AB)^T = B^T A^T$.

Pf: (a) $\left((AB)^T\right)_{ij} = AB_{ji}$ = < jth row of A, ith col of B >

Say AB is $m \times n$

= < jth col of A^T, ith row of B^T >

= < ith row of B^T, jth col of A^T >

$= \left(B^T A^T\right)_{ij}$ $\forall\ 1 \le i \le n,\ 1 \le j \le m$

(b). Check $(A^{-1})^T \cdot A^T = A^T \cdot (A^{-1})^T = I$ using (a).

(c) & (d): easier. Ex.

Next time: Midterm I.