Math 2135. Lecture 15.   Midterm I : Friday   09.27.2021.

Last time: · Properties of matrix multiplication:

— No commutativity or cancellation for multiplication: in general, $AB \neq BA$; $AB = AC \not\Rightarrow B = C$; $AB = CB \not\Rightarrow A = C$.

— Other obvious generalizations of scalar arithmetic do work (e.g. $A(B+C) = AB + AC$)

— Multiplying with a diagonal matrix $D = \mathrm{diag}_n(a_1, \ldots, a_n)$:

left mult by D ($A \mapsto DA$) scales the rows of a matrix A;
right ---- ---- ($A \mapsto AD$) ------ cols of ---- ---- .

e.g.
$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 9 \\ 0 & 4 & -4 \end{bmatrix}; \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 4 & 15 \\ 0 & 4 & -5 \end{bmatrix}$$

Ex. If A, D are $n \times n$ and D is diag. Is $AD = DA$ in general?

e.g. (identity matrix) We call the matrix $I_n := \mathrm{diag}(1, 1, \ldots, 1) = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$ the $n \times n$ identity matrix. If A is an $m \times n$ matrix, then $I_m A = A = A I_n$.

Today: · associativity of matrix multiplication, via linear maps.

· more matrix operation(s): inversion, transposition, powers.

1. Associativity.

Q: Why is matrix mult. defined this way?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{l} \nearrow = \begin{bmatrix} 5 & 10 \\ 15 & 22 \end{bmatrix} \text{ by the "row-col rule".} \\ \searrow \neq \begin{bmatrix} 1\cdot 1 & 2\cdot 2 \\ 3\cdot 3 & 4\cdot 4 \end{bmatrix} \end{array}$$

Answer: So that it's compatible with composition ... (next page)

Setting: Let A, B be $m\times n$ and $n\times p$ matrices and let T_A, T_B be the associated linear maps $T_A: \mathbb{R}^n \to \mathbb{R}^m$, $T_A(x) = A\cdot x \;\forall x \in \mathbb{R}^n$

$T_B: \mathbb{R}^p \to \mathbb{R}^n$, $T_B(x) = Bx \;\forall x \in \mathbb{R}^p$.

It now makes sense to consider the composition

$$T_A \circ T_B : \quad \mathbb{R}^m \xleftarrow{T_A} \mathbb{R}^n \xleftarrow{T_B} \mathbb{R}^p$$

Fact: The composition of two linear maps is another linear map.

Thus, $T_A \circ T_B : \mathbb{R}^p \to \mathbb{R}^m$ has a standard matrix M.

$$\left(T_A \leftrightarrow A, \quad T_B \leftrightarrow B, \qquad T_A \circ T_B \leftrightarrow \overset{?}{M} \right)$$

Thm: In the above setting, the standard matrix of $T_A \circ T_B$ is exactly AB, i.e., we have $T_A \circ T_B(x) = (AB)x \quad \forall x \in \mathbb{R}^p$.

(the standard matrix of the comp is the product of the standard matrices)

Pf: Write $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ for $x \in \mathbb{R}^p$, and write $B = [b_1 | b_2 | \cdots | b_p]$.

Then $T_A \circ T_B(x) = T_A\Big(T_B(x)\Big) = T_A(Bx) = A \cdot (\ Bx\ )$

$$= A \cdot \left(x_1 b_1 + \cdots + x_p b_p \right) \xrightarrow[\text{w/ } A]{\text{lin. of mult}} x_1 (Ab_1) + x_2 (Ab_2) \cdots + x_p (Ab_p)$$

$$= [Ab_1 | Ab_2 | \cdots | Ab_p] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \underbrace{[Ab_1 | Ab_2 | \cdots | Ab_p]}_{AB} \cdot x = (AB)x.$$

$\square$.

We can deduce the associativity of mat. mult from the theorem:

Let A, B, C be $m\times n$, $n\times p$, $p\times q$ matrices, and consider the maps

$T_A: \mathbb{R}^n \to \mathbb{R}^m$, $T_B: \mathbb{R}^p \to \mathbb{R}^n$, $T_C: \mathbb{R}^q \to \mathbb{R}^p$ given by left

mult. by A, B, C, respectively.

By the theorem, $(AB)C$ is the standard matrix of $(T_A \circ T_B)\circ T_C$

and $A(BC)$ is .... $T_A \circ (T_B \circ T_C)$

But the maps $(T_A \circ T_B)\circ T_C$ and $T_A \circ (T_B \circ T_C)$ are equal because

composition of functions is associative. It follows that $(AB)C = A(BC)$. $\square$.

E.g. (for the Thm). Take $m=n=p=2$. Consider the linear maps.

$T_A = T : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix}$ $\to$ refl. w.r.t to the x axis, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$T_B = S : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$ $\to$ refl. w.r.t. to the line $y=x$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

compose :

$T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \end{bmatrix} \overset{S}{\mapsto} \begin{bmatrix} y \\ x \end{bmatrix} \overset{T}{\mapsto} \begin{bmatrix} y \\ -x \end{bmatrix}$

linear. the standard matrix for $T \circ S = T_A \circ T_B$ is

$\begin{bmatrix} T\circ S(e_1) & T\circ S(e_2) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}}$

What about AB?

$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}}$.

This is indeed AB; agrees w/ the Thm.

2. Matrix Inversion

Def: An $n\times n$ matrix A is called <u>invertible</u> if there is an $n\times n$ matrix B s.t. $AB = I_n = BA$. In this case, we call B the inverse of A and denote it by A^{-1}.

Note: We only discuss invertibility for square matrices.

Eg. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $B = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ is an inverse of A.

Pf: $AB = \begin{bmatrix} 1\cdot(-2)+2\cdot\frac{3}{2} & 1\cdot 1+2\cdot(-\frac{1}{2}) \\ 3\cdot(-2)+4\cdot\frac{3}{2} & 3\cdot 1+4\cdot(-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ex: Check that $BA = I_2$.

Invertibility allows cancellation

- $AB = AC$, A is inv $\Longrightarrow$ $B = C$.

$$\Downarrow \qquad\qquad \Uparrow$$

$$(A^{-1})AB = A^{-1}(AC) \Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I \cdot B = I \cdot C$$

Similar to $2 \cdot x = 2 \cdot y \Rightarrow \frac{1}{2} \cdot (2 \cdot x) = \frac{1}{2} \cdot (2 \cdot y) \Rightarrow x = y$.

- $Ax = b$, A inv $\Longrightarrow$ $x = A^{-1}b$.

Pf: Ex.

Next time:
- more on inversion and cancellation
- invertibility of 2×2 matrices.
- transposes & powers
- midterm review.