Math 2135. Lecture 14.

Midterm I: next Friday, in class.

09. 24. 2021.

Last time:

- geometric transformations in $\mathbb{R}^2$.
- matrix operations: addition, scaling, multiplication.

Today:

- properties and non-properties of matrix operations.

Warm-up: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Let $k \in \mathbb{R}$ be an arbitrary scalar.

Compute $A(kB) = k(AB)$, $AB \neq BA$.

Soln:

$$A(kB) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2k & 3k \\ 4k & 5k \end{bmatrix} = \begin{bmatrix} 1\cdot 2k + 2\cdot 4k & 1\cdot 3k + 2\cdot 5k \\ 3\cdot 2k + 4\cdot 4k & 3\cdot 3k + 4\cdot 5k \end{bmatrix}$$

scalar arithmetic

$$k(AB) = k \underbrace{\begin{bmatrix} 1\cdot 2 + 2\cdot 4 & 1\cdot 3 + 2\cdot 5 \\ 3\cdot 2 + 4\cdot 4 & 3\cdot 3 + 4\cdot 5 \end{bmatrix}}_{} = \begin{bmatrix} k(1\cdot 2 + 2\cdot 4) & k(1\cdot 3 + 2\cdot 5) \\ k(3\cdot 2 + 4\cdot 4) & k(3\cdot 3 + 4\cdot 5) \end{bmatrix}$$

$AB \neq BA$. $\leftarrow$ $AB = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 3 & 2\cdot 2 + 3\cdot 4 \\ 4\cdot 1 + 5\cdot 3 & 4\cdot 2 + 5\cdot 4 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ 19 & 28 \end{bmatrix}$$

Prop 1: Let A, B, C be $m \times n$ matrices and let 0 be the $m \times n$ matrix

(no mult.) where all entries are 0. Let $r, s \in \mathbb{R}$. Then

(1) (+ is associative) $(A+B)+C = A+(B+C)$.

(2) (+ is commutative) $A+B = B+A$.

(3) (0 is the additive identity:) $A+0 = A = 0+A$.

(4) (scaling distributes over matrix addition) $r(A+B) = rA + rB$.

(5) (. . . . . scalar addition) $(r+s)A = rA + sA$

(6) $r(sA) = (rs)A$

E.g. (4). distributivity of scaling over mat. $+$:

$$r=2, \quad A=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}.$$

$$r(A+B) = 2\cdot\begin{bmatrix} 2+1 & 1+0 \\ -1+5 & 3+0 \end{bmatrix} = \begin{bmatrix} 2\cdot(2+1) & 2\cdot(1+0) \\ 2\cdot(-1+5) & 2\cdot(3+0) \end{bmatrix}$$

scalar arithmetic $\big\Vert$

$$rA+rB = \begin{bmatrix} 2\cdot 2 & 2\cdot 1 \\ 2\cdot(-1) & 2\cdot 3 \end{bmatrix} + \begin{bmatrix} 2\cdot 1 & 2\cdot 0 \\ 2\cdot 5 & 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 2+2\cdot 1 & 2\cdot 1+2\cdot 0 \\ 2\cdot(-1)+2\cdot 5 & 2\cdot 3+2\cdot 0 \end{bmatrix}$$

Pf of Prop 1 (sketch): The properties hold because they hold for scalar and addition & scaling of matrices are defined coordinatewise.

e.g. $(A+B)=(B+A)$ because $\forall\ 1\le i\le m,\ 1\le j\le n,$

$$(A+B)_{ij} = \underbrace{A_{ij}+B_{ij}}_{\text{scalars}} = B_{ij}+A_{ij} = (B+A)_{ij}.$$

Properties (and non-properties) involving multiplications. (so AB is defined)

Let A be an $\underline{m\times n}$ matrix and let B be an $\underline{n\times p}$ matrices. Let $c \in \mathbb{R}$.

(0). BA may not be defined. (e.g. $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ $B=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$, $n=2$.)

(1) Mult. is not coordinatewise: in general $(AB)_{ij} \neq A_{ij}B_{ij}$ (e.g. $A=B=\begin{bmatrix}1&2\\3&4\end{bmatrix}$)

(2) No commutativity: Even if AB and BA both make sense, they need not be equal.

(e.g. Page 1: $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $B=\begin{bmatrix}2&3\\4&5\end{bmatrix}$.

Last time: $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ $B=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$.)

(3). No cancellation: $AB=AC$, $A\neq 0$ $\not\Rightarrow$ $B=C$. (e.g. $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $B=\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $C=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$)

Similarly, $BA=CA$, $A\neq 0$ $\not\Rightarrow$ $B=C$ → Ex: find e.g.

Prop 2. Let A, B, C be $m\times n$, $n\times p$, $p\times q$ matrices, respectively.

(1) (mat. mult. is associative) $(AB)C = A(BC)$. $\longrightarrow$ Not obvious!

E.g. $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

$$(AB)C = \underline{\begin{bmatrix} 3 & 4 \\ 11 & 16 \end{bmatrix}} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 29 \\ 86 & 113 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \underline{\begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}} = \begin{bmatrix} 22 & 29 \\ 86 & 113 \end{bmatrix}$$

12) ($\times$ and $+$ are compatible: mult. distributes over addition)

If B' is an $n\times p$ matrix, then $A(B+B') = AB + AB'$.

Similarly, if A' is an $m\times n$ matrix, then $(A+A')B = AB + A'B$.

13) ($\times$ and scaling are compatible)

$$(cA)\cdot B = c(AB) = A\cdot(cB)$$

Pf sketch: 1): next time. 2) & 3) follow from the corresponding arithmetic laws for scalars.

2): $\forall\, 1\le i\le m,\ 1\le j\le p$, $\left[A(B+B')\right]_{ij} = \sum_{k=1}^{n} A_{ik}(B+B')_{kj} = \sum_{k=1}^{n}\left(A_{ik}B_{kj} + A_{ik}B'_{kj}\right)$

$$= \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}B'_{kj} = [AB]_{ij} + [AB']_{ij} = [AB+AB']_{ij}.$$

3): Similar to 2). EX.

Prop 3. (multiplication w/ diagonal matrices "scales rows or cols")

Let $D = \text{diag}_n(a_1, a_2, \dots, a_n)$ be the square and diagonal matrix

$$\begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

e.g. $\text{diag}_3(2,1,4) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

If B is an $n \times p$ matrix, left-multiply B with D (DB) scales the ith row of B by a_i $\forall 1 \le i \le n$. If A is $m \times n$, then right-multiplying A by D scales the ith col of A $\forall 1 \le i \le n$.

Next time:
- pf of $(AB)C = A(BC)$
- inverse of matrices.

E.g. $n=2$, $m=3$, $p=4$.

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 8 & 4 & 0 & 4 \end{bmatrix};$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 5 \\ 3 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}}_{D} = \begin{bmatrix} 3 & -4 \\ 0 & 20 \\ 9 & 16 \end{bmatrix}.$$