Math 2135. Lecture 13.

09.22.2021.

Last time:

- Two sets of equivalences: if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with standard matrix A (so $T(x) = A \cdot x \ \forall x \in \mathbb{R}^n$; $A = [T(e_1) | \cdots T(e_n)]$).

Then TFAE: (1) T is surj. (2) $\operatorname{Im} T = \mathbb{R}^m$.

(3) The cols of A span $\mathbb{R}^m$. (4) $EF(A)$ has no zero row.

And TFAE: (1) T is inj. (2) $\ker T = \{0\}$.

(3) The cols of A are lin. ind. (4) all cols in A are pivot.

- Geometric transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$:

(i) reflection across the x-axis / y-axis; (ii). projection onto the x-axis / y-axis.

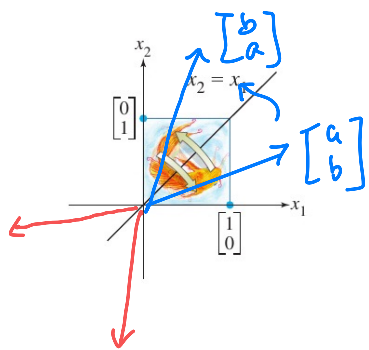
Today: · finish Ch 1: more geom. maps

· start Ch. 2: matrix operations (addition, scaling, multiplication)

1. More geometric maps

(a) Warm-up: reflection across the line $x_2 = x_1$.

Reflection through the line $x_2 = x_1$



$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Ex: explain this geometrically.

Formula: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$

Standard matrix A: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

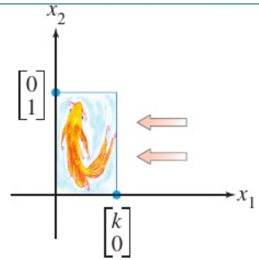
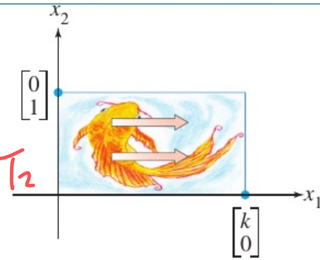
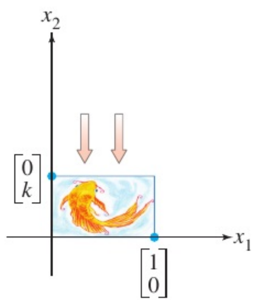
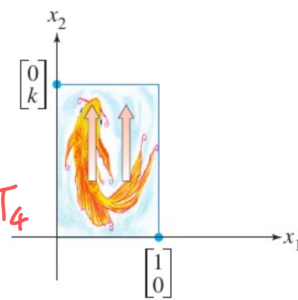
EF(A): $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Surj? : no zero row in EF(A), So the map is surj.

inj? : all cols in EF(A) are pivot, so the map is inj.

(b) Contractions and expansions (assume $k \neq 0$)

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	T_1  $0 < k < 1$	T_2  $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	T_3  $0 < k < 1$	T_4  $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Formulas: $\begin{bmatrix} x \\ y \end{bmatrix} \xmapsto{T_1, T_2} \begin{bmatrix} kx \\ y \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} \xmapsto{T_3, T_4} \begin{bmatrix} x \\ ky \end{bmatrix}$

Standard matrices:

$T_1, T_2 : \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$; $T_3, T_4 : \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

→ EF ←

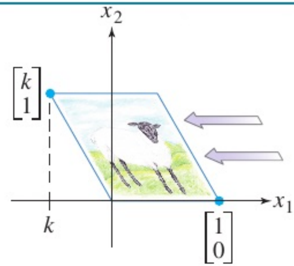
Surj? they all are.

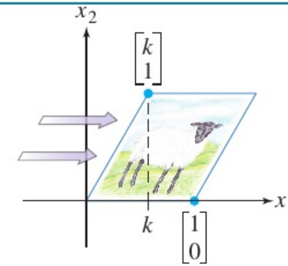
$T_1 - T_4$.

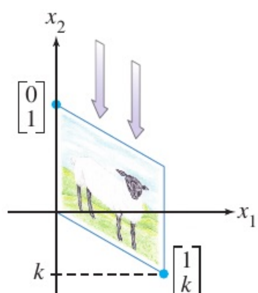
inj? they all are

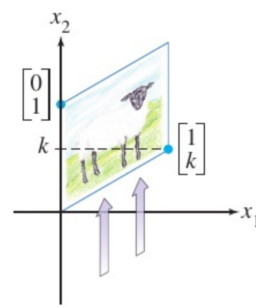
1c) Shears

TABLE 3 **Shears**

<table>
<tr><th>Transformation</th><th colspan="2">Image of the Unit Square</th><th>Standard Matrix</th></tr>
<tr><td>Horizontal shear</td><td>

$k < 0$</td><td>

$k > 0$</td><td>$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Vertical shear</td><td>

$k < 0$</td><td>

$k > 0$</td><td>$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$</td></tr>
</table>

Ex: Explain why these four maps are all both surj and inj.

d) Rotations

Ex: $\forall \varphi \in [0, 2\pi)$, R_φ is surj and inj.

Let $\varphi \in [0, 2\pi)$ be an angle and let $R_\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that rotates every vector in $\mathbb{R}^2$ by φ.

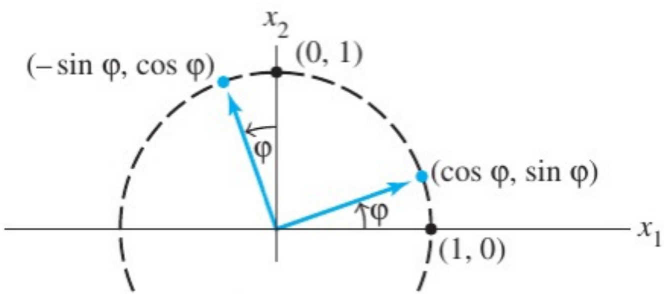


FIGURE 1 A rotation transformation.

Note: R_φ is a linear map!

- Directly finding the formula for R_φ is hard.

- However, since we know R_φ is linear, we can find its formula via its standard matrix A: $A = [R_\varphi(e_1) | R_\varphi(e_2)] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, therefore

$$R_\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\varphi \cdot x - \sin\varphi \cdot y \\ \sin\varphi \cdot x + \cos\varphi \cdot y \end{bmatrix}.$$

2. Matrix operations

Notation: For each $m \times n$ matrix A and all indices $1 \leq i \leq m, 1 \leq j \leq n$, we denote the entry in the ith row, jth col. of A by A_{ij}, and write $A = [A_{ij}]$.

(i) Addition and scaling.

Matrix addition and scaling are defined coordinatewise (just as for vectors).

More precisely, if A, B are both $m \times n$ matrices and $c \in \mathbb{R}$.

Then $A+B$ is the $m \times n$ matrix defined by

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

and cA is the $m \times n$ matrix defined by $(cA)_{ij} = c \cdot A_{ij}$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}$.

Then

$$A + B = \begin{bmatrix} 1+2 & 2+1 & 3+0 \\ 4+0 & 5-1 & 6-2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 2+1 & 1+2 & 0+3 \\ 0+4 & -1+5 & -2+6 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$$

$$3 \cdot B = \begin{bmatrix} 3\cdot 2 & 3\cdot 1 & 3\cdot 0 \\ 3\cdot 0 & 3\cdot(-1) & 3\cdot(-2) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 \\ 0 & -3 & -6 \end{bmatrix}$$

$$A - 2\cdot B = \begin{bmatrix} 1-2\cdot 2 & 2-2\cdot 1 & 3-2\cdot 0 \\ 4-2\cdot 0 & 5-2\cdot(-1) & 6-2\cdot(-2) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 \\ 4 & 7 & 10 \end{bmatrix}.$$

Theme: We'll be especially interested in the properties of each operation and of the interactions of the operations.

e.g. $A + B = B + A$. $A(B + C) = A\cdot B + A\cdot C$ for all matrices A, B, C for which the expressions make sense.

(ii) Matrix - matrix multiplication

Let A, B be matrices. We can define the product AB when (and only when) # cols of A = # rows of B.

When this is the case, the product AB is defined as follows:

Def: Let A be an $m \times n$ matrix and let B be a $n \times p$ matrix.

Write $B = [v_1 | v_2 | \cdots | v_p]$. We define AB to be the matrix

$$AB = A[v_1 | \cdots | v_p] \overset{*}{=} [\underbrace{A \cdot v_1} | \underbrace{A \cdot v_2} | \cdots | \underbrace{Av_p}].$$

matrix-vec. product, $Av_i \in \mathbb{R}^m$ $\forall i$.

(So · AB is $m \times p$.

· if $p = 1$, ie., B is a column vector, then $*$ recovers our old formula for matrix vec products.)

E.g. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, $B = \begin{array}{c} \left[\begin{array}{c|c} 2 & 1 \\ 1 & 0 \\ 1 & 1 \end{array}\right] \\ \begin{array}{cc} v_1 & v_2 \end{array} \end{array}$, $B_v = \begin{array}{c} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ v_1 \end{array}$.

Then $AB = \left[Av_1 \middle| Av_2 \right] = \left[\begin{array}{c|c} 1\cdot 2 + 0\cdot 1 + 1\cdot 1 & \\ 1\cdot 2 - 1\cdot 1 + 0 & \cdots \end{array}\right] = \left[\begin{array}{c|c} 3 & 2 \\ 1 & 1 \end{array}\right]$. 2×2

$AB_v = \left[Av_1 \right] = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$AB \neq BA$

$BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{array}{c} \left[\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & -1 & 0 \end{array}\right] \\ \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \end{array} = \left[Bw_1 \middle| Bw_2 \middle| Bw_3 \right] = \left[\begin{array}{c|c|c} 3 & & \\ 1 & \cdots & \cdots \\ 2 & & \end{array}\right]$ 3×3

BB_v is not defined.

If $A = \begin{bmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \\ \hline r_m \end{bmatrix}$ and $B = \left[c_1 \middle| \cdots \middle| c_p \right]$, then $\forall\, 1 \leq i \leq m,\ 1 \leq j \leq n$,

$(AB)_{ij}$ = inner product of Row i of A and Col. j of $B = \langle r_i, c_j \rangle$.

Note: Unlike addition & scaling, matrix multiplication IS not defined coordinatewise! (even when such def is possible)

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

- Check that AB, BA are both 2×2 but not equal.
- Check that $(AB)C = A(BC)$.

Next time:

- more example computations.
- properties / non-properties of matrix operations.