Math 2135. Lecture 12.

09. 20. 2021.

Last time:

- linear maps ($\mathbb{R}^n \to \mathbb{R}^m$) $\equiv$ matrix mult.

$$\{\text{linear maps } \mathbb{R}^n \to \mathbb{R}^m\} \xleftrightarrow{\text{bij.}} \{m \times n \text{ matrices}\}$$

$$(\text{the map } T_A : x \mapsto Ax) \longleftarrow\!\shortmid \; A$$

$$T \longmapsto \text{the standard matrix } A_T.$$

- Four notions associated to lin maps ($T: \mathbb{R}^n \to \mathbb{R}^m$ lin.)

image (range), kernel ("zero set"), surjectivity ("onto"), injectivity ("one-to-one").

Today.

- Connecting $\operatorname{Im} T$, $\ker T$, surj, inj, spans, lin. ind, EF.

- Geometric transformations of $\mathbb{R}^2$

E.g. Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ 4x+2y \end{bmatrix}$.

Find the standard matrix A_T of T, $\operatorname{Im} T$, $\operatorname{Ker} T$; determine if T is surj, inj.

① ② ③ ④ ⑤

Sol$_n$:

$$A_T = \left[T(e_1) \middle| T(e_2) \right] = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}.$$

$$\operatorname{Im} T = \left\{ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2x+y \\ 4x+2y \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} (2x+y) \\ 2(2x+y) \end{bmatrix} : \underset{\mathbb{R}}{x, y} \right\}$$

$$= \left\{ (2x+y) \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x, y \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \quad (\neq \mathbb{R}^2)$$

$$\ker T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (2x+y) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x+y = 0 \right\}$$

Since $\operatorname{Im} T \neq \mathbb{R}^2$, T is not surj.

$$= \left\{ \begin{bmatrix} -y/2 \\ y \end{bmatrix} \right\} = y \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} : y \in \mathbb{R} \Big\}.$$

Since both $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ have output $\vec{0}$, T is not inj.

Note: Surj is related to image; Inj is related to kernel. more precisely.

Thm 1. (image, surjectivity, and spanning problems)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin map. Let A_T be the standard matrix of T. $(T(x) = A_T \cdot x \ \forall x \in \mathbb{R}^n)$.

TFAE: (1). T is surj.

(2) $\operatorname{Im} T = \mathbb{R}^m$.

(3). The cols of A_T span $\mathbb{R}^m$.

(4). (EF) $EF(A_T)$ has no zero rows.

Pf: (1) $\Leftrightarrow$ (2) by def. of surj. maps; (3) $\Leftrightarrow$ (4) by previous theorems.

So it suffices to prove that (2) $\Leftrightarrow$ (3). The last equivalence follows from the fact that

$$\operatorname{Im} T = \{ T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n \} = \Big\{ \underbrace{A_T}_{[w_1 | \cdots | w_n]} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} : c_1, \cdots, c_n \in \mathbb{R} \Big\} = \Big\{ \sum_{i=1}^{n} c_i w_i \,\Big|\, c_1, \cdots, c_n \in \mathbb{R} \Big\} = \operatorname{Span}(\text{cols of } A_T).$$

$\square$

Thm 2. (kernel, injectivity, and lin ind.) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. map and let A_T be its standard matrix. Then TFAE:

(1) T is inj.

(2) $\ker T = \{\vec{0}\}$.

(3) The cols of A_T are lin. ind.

(4) Every col in A_T (or $EF(A_T)$) is pivot.

Pf: (sketch) Again, (3) $\Leftrightarrow$ (4) by previous results.

(2) $\Leftrightarrow$ (3): Note that $\ker T = \{ \vec{v} \in \mathbb{R}^n \mid T(v) = \underbrace{A_T \cdot v = 0}_{[w_1 | \cdots | w_n]} \} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1 w_1 + \cdots + x_n w_n = 0 \right\}$,

that is, $\ker T$ is the soln set of the vec. eq. $x_1 w_1 + \cdots + x_n w_n \overset{(*)}{=} 0$.

It follows that (2) holds $\Leftrightarrow$ the only soln of $(*)$ is the trivial one $\Leftrightarrow$ (3) holds.

It remains to prove (1) $\Leftrightarrow$ (2):

(1) $\Rightarrow$ (2): if T is inj, then only $\vec{0}$ can be sent to $\vec{0}$, so $\ker T = \{\vec{0}\}$. ✓

(2) $\Rightarrow$ (1): Suppose $\ker T = \{0\}$. We want to prove that T is inj.

So assume $T(v) = T(v')$ for $v, v' \in \mathbb{R}^n$.

Then $T(v) - T(v') = 0$,

so by linearity of T, $T(v - v') = 0$.

But $\ker T = \{0\}$, so $v - v' = 0$ and $v = v'$.

It follows that T is inj, as desired.

$\square$

Example applications: • Consider the linear map $T: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x-y \\ x+y \\ 3y \end{bmatrix}$.

(1) Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\operatorname{Im} T$? $\Leftrightarrow$ Is there an $\begin{bmatrix} x \\ y \end{bmatrix}$ s.t. $\begin{bmatrix} 2x-y \\ x+y \\ 3y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

$\Downarrow$

Spanning problem! Ex.

(2) Is T surj? inj?

$$A_T = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 3 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}}_{EF}$$

$EF(A_T)$ has a zero row, so T is not surj.

$EF(A_T)$ has only pivot cols, so T is inj.

• Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Prove that

(1) If $n > m$, then T cannot be inj.

(2) If $n < m$, then T cannot be surj.

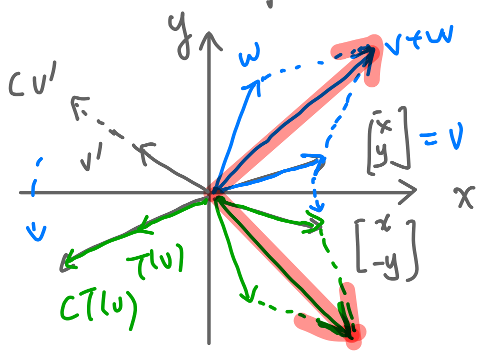
Pf: Ex. (Hint: use Thms 1, 2 & Page 7 of Lecture 8.)

Geometric linear maps on $\mathbb{R}^2 \to \mathbb{R}^2$

First example. (reflection across the x-axis)

Consider the map $Ref_x : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The map Ref_x is linear. it is called "reflection across the x-axis"

(w.r.t.) with respect to



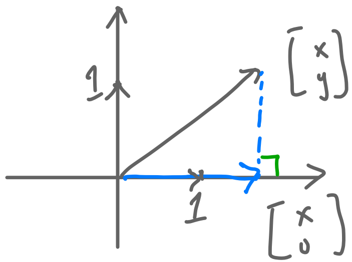
in pictures

$$Ref_x(v+w) = Ref_x(v) + Ref_x(w)$$

$$Ref_x(cv) = c\, Ref_x(v).$$

. T has standard matrix $\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{EF}$; it is surj & inj.

Example 2. Projection onto the x-axis.



The map $T = \mathrm{Proj}_x : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

is also a linear map.

We call T <u>the projection onto the x-axis</u>.

The standard matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $\leftarrow \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\nwarrow \left[T(e_1) \,\middle|\, T(e_2) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

T is not surj and not inj either.

Next time:
- more geometric maps
- matrix multiplication.