Math 2135. Lecture 11. Midterm 1. Fri. Oct. 01. 09.17.2021

Last time. · Def of linear maps: a map $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

(1) $T(v+w) = T(v) + T(w)$ and (2) $T(cv) = cT(v)$ $\forall v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

· Properties of linear maps: say $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

— (0 to 0) $T(\vec{0}) = \vec{0}$.

— (new from old) if $T(v_1), \cdots, T(v_k)$ are known, then $T(v)$ can be computed $\forall v \in \text{Span}\{v_1, \cdots, v_k\}$: $v = c_1v_1 + \cdots + c_kv_k \Rightarrow T(v) = T(c_1v_1 + \cdots + c_kv_k)$

$= c_1 T(v_1) + \cdots + c_k T(v_k)$

— ("component") A map $S: \mathbb{R}^n \to \mathbb{R}^m$, $v \mapsto \begin{bmatrix} S_1(v) \\ \vdots \\ S_m(v) \end{bmatrix}$ is linear iff S_i is linear $\forall i$.

· Examples and formulas of linear maps: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ 2x-3y \end{bmatrix}$ ✓ linear.

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \sin x \\ x^2+y \end{bmatrix}$ ✗ not linear.

Claim: A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear iff there is a matrix A s.t. $T(v) = A \cdot v$ $\forall v \in \mathbb{R}^n$.

Today: · proof of the claim

· notions associated to linear maps.

1. Linear maps $\equiv$ matrix multiplication

e.g. $m=3$, $n=2$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $c = 5$.

Note that for any $m \times n$ matrix A and $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

we have (i) $\underbrace{A(v+w)}_{\text{in } \mathbb{R}^m} = \underbrace{Av + Aw}_{\text{in } \mathbb{R}^m}$ and (ii) $A(cw) = cAw$

pf: Suppose $A = [a_{ij}]$ (ijth entry being a_{ij}) and $v = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $w = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

For (i), we note that $\forall\ 1 \le j \le m$,

$$[A(v+w)]_j = \sum_{k=1}^{n} A_{jk} (v+w)_k = \sum_{k=1}^{n} A_{jk} (v_k + w_k)$$

$$= \sum_{k=1}^{n} A_{jk} v_k + \sum_{k=1}^{n} A_{jk} w_k = [Av]_j + [Aw]_j = [Av + Aw]_j.$$

so $A(v+w) = Av + Aw$. (ii) can be proved similarly.

Prop 1. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a map given by $T(x) = Ax$ $\forall x \in \mathbb{R}^n$, where A is a matrix.

Then T is linear.

Pf: Let $v, w \in \mathbb{R}^n$, and let $c \in \mathbb{R}$.

(1) $T(v+w) = A(v+w) \overset{(i)}{=} Av + Aw = T(v) + T(w)$.

(2) $T(cv) = A(cv) \overset{(ii)}{=} cAv = cT(v)$.

By (1) and (2), T is linear. $\square$

Eg. Prove that the map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x-y \\ 3x \end{bmatrix}$ is linear.

Pf: Note that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ 3x \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}}_{\text{"}A\text{"}} \begin{bmatrix} x \\ y \end{bmatrix}$;

that is, T is mult by the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$,

so T is a linear map by Prop 1.

Thm 2. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there exists a unique matrix A_T s.t. $T(x) = A_T \cdot x \;\; \forall\, x \in \mathbb{R}^n$.

This unique matrix is

$$A_T = \left[\, T(e_1) \,\middle|\, T(e_2) \,\middle|\, \cdots \,\middle|\, T(e_n) \,\right]$$

where $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is the so-called standard basis of $\mathbb{R}^n$.

Def: The matrix A_T associated to the linear T is called the standard matrix of T.

e.g.: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x - y \\ 3x \end{bmatrix}$

earlier observation:

$$\begin{bmatrix} 2x - y \\ 3x \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

the theorem: $A_T = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \middle| T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \left[\begin{matrix} 2 \\ 3 \end{matrix} \middle| \begin{matrix} -1 \\ 0 \end{matrix} \right]$,

(deterministic)

Pf: Write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ for $\vec{x} \in \mathbb{R}^n$. Note that $\vec{x} = x_1 e_1 + \cdots + x_n e_n$.

Since T is linear,

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n)$$

$$= x_1 \underbrace{T(e_1)}_{\in \mathbb{R}^m} + \cdots + x_n \underbrace{T(e_n)}_{\in \mathbb{R}^m} \quad \text{by linearity}$$

$$= \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{by def of matrix-vect. product}$$

$$= A_T \cdot \vec{x}$$

(uniqueness): EX.

$\square$.

Remarks:

- Prop 1 and Thm 2 combine to prove our claim from last time:

  "linear maps $\equiv$ matrix multiplications".

  In particular, if $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(v) = \begin{bmatrix} T_1(v) \\ \vdots \\ T_m(v) \end{bmatrix} \forall v \in \mathbb{R}^n$,

  then T is linear iff each $T_i(v)$ is a lin. comb of the coordinates of v.

- Later, we will study linear maps via their standard matrices.

2. Image, Kernel, Injectivity and Surjectivity

Def. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

(1) The <u>image</u> of T is defined to be the set

$$\operatorname{Im} T := \{ T(v) \mid v \in \mathbb{R}^n \}.$$

(2) The <u>kernel</u> of T is the set

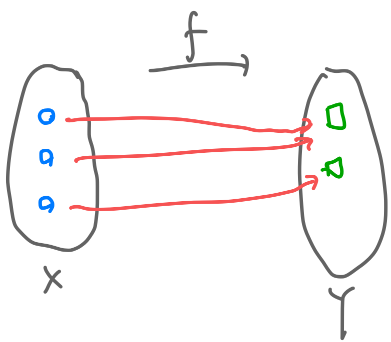
$$\operatorname{Ker} T := \{ v \in \mathbb{R}^n \mid T(v) = 0 \}.$$

<u>Note:</u> Since $T(v) = A_T \cdot v \ \forall v \in \mathbb{R}^n$ where A_T is the standard matrix, $\ker T$ is just the soln set of the hom eq. $A_T \cdot x = 0$.

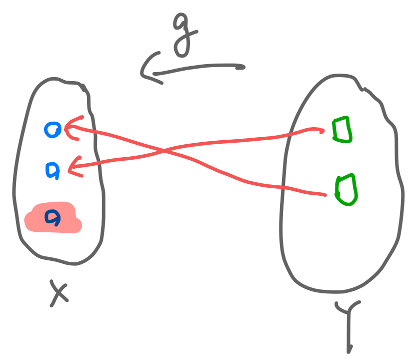
(3) We say T is <u>surjective</u> (surj.) or <u>onto</u> if every elt in $\mathbb{R}^m$ is an output of T, i.e., if $\operatorname{Im} T = \mathbb{R}^m$.

(4) We say T is <u>injective</u> (inj.) or <u>one-to-one</u> if every elt in $\mathbb{R}^n$ is the output of at most one input, i.e., if "$T(v) = T(v')$ for $v, v' \in \mathbb{R}^n \Rightarrow v = v'$".

<u>Note:</u> By definition,



surj; not inj.



not surj; inj.

E.g. Consider the map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ 4x+y \end{bmatrix}$.

(0) Find the standard matrix of T.

(1) Find the image of T.

(2) Find the kernel of T (in parametric vector form).

(3) Determine whether T is surj.

(4) Determine whether T is inj.

Ex: Try (0) – (4)

Next time:

- more examples.
- "T is inj $\iff \ker T = \{0\}$"
- connecting surj/inj to span/lin ind.
- geom maps in $\mathbb{R}^2$ & $\mathbb{R}^3$