Math 2135. Lecture 10.

09. 15. 2021.

Last time:

- the ▱ law for vect. addition
- soln sets of homogeneous vs. non-homogeneous matrix equations.

Today:

- linear transformations

1. Definition A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear map / transformation</u> if

(1) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$

"T respects / commutes with +"

and (2) $T(c \cdot \vec{v}) = c \cdot T(\vec{v}) \quad \forall c \in \mathbb{R}, \vec{v} \in \mathbb{R}^n$.

"T respects / commutes with scaling".

Remarks:

① Later, we'll call generalize the last def to all maps between so called "vector spaces": a map $T: V \to W$ where V, W are vector spaces is <u>linear</u> if it respects addition and scaling.

E.g. The set $P_n = \{$ polynomials in $\mathbb{R}[t]$ of degree $\le n \}$ is a vector space (addition and scaling makes sense for poly.) $\forall n$.

The formal differentiation map $d : P_3 \longrightarrow P_2$, $f \mapsto f'$!

(e.g. $x^2 \mapsto 2x$)

is linear because So we've all seen linearity!

$$(f+g)' = f' + g' \quad \text{and} \quad (cf)' = c \cdot f' \quad \forall f, g \in P_3 \text{ and } c \in \mathbb{R}.$$

② **Prop 0**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $\underset{\mathbb{R}^n}{T(\vec{0})} = \underset{\mathbb{R}^m}{\vec{0}}$.

pf: Since $\vec{0} = \vec{0} + \vec{0}$ in $\mathbb{R}^n$, we have

$$\underline{T(\vec{0})} = T(\vec{0} + \vec{0}) \overset{(1)}{=} T(\vec{0}) + T(\vec{0}) = 2\,\underline{T(\vec{0})} \qquad (*)$$

(e.g. If $m=3$ & $T(\vec{0}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then $(*)$ says $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$)

It follows that $T(\vec{0}) = \vec{0}$ in $\mathbb{R}^m$.

2. "New from old".

Since linear maps respect $+$ and scaling, they respect lin. comb:

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $\forall c_1, c_2, \dots, c_k \in \mathbb{R}, v_1, \dots, v_k \in \mathbb{R}^n$,

we have

$$T(c_1 v_1 + \cdots + c_k v_k) \overset{(1)}{=} T(c_1 v_1) + \cdots + T(c_k v_k) \overset{(2)}{=} c_1 T(v_1) + \cdots + c_k T(v_k)$$

Point: If T is linear and we know $T(v_1), \dots, T(v_k)$, then we know $T(v)$ for all $v \in \operatorname{Span}\{v_1, \dots, v_k\}$.

Examples

(1) Say we have a lin. map $T: \mathbb{R}^2 \to \mathbb{R}^3$ with

$$\underbrace{T\left(\overset{\vec{v}_1}{\begin{bmatrix}1\\2\end{bmatrix}}\right)}_{\text{known}} = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad \underbrace{T\left(\overset{\vec{v}_2}{\begin{bmatrix}2\\3\end{bmatrix}}\right)}_{\text{known}} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then

$$T\left(\begin{bmatrix}3\\5\end{bmatrix}\right) = T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\8\end{bmatrix}\right) = T(\vec{v}_1 + 2\vec{v}_2) = T(\vec{v}_1) + 2T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} + 2\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = T(\vec{v}_1 - \vec{v}_2) \boxed{= T(\vec{v}_1 + (-1)\cdot\vec{v}_2) = T(\vec{v}) + (-1)T(\vec{v}_2)}$$

$$= T(\vec{v}_1) - T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

(2). Say we have a lin map $\mathcal{U}: \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\mathcal{U}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathcal{U}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathcal{U}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

Then for an arbitrary elt $\vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$, since

$$\vec{v} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix},$$

we have

$$\mathcal{U}(\vec{v}) = \mathcal{U}\left(a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = a\,\mathcal{U}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + b\,\mathcal{U}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + c\,\mathcal{U}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}a-c\\2a+b-c\\3a-c\end{bmatrix}. \quad \square$$

3. Formulas of linear maps

Examples. Determine if the following maps $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

(1). $T: \mathbb{R} \to \mathbb{R}$, $T(x) = 2x \quad \forall x \in \mathbb{R}$ $\quad (x \mapsto 2x)$

Let's check the axioms.

(i)
$$\left.\begin{aligned} T(x+y) &= 2(x+y) = 2x+2y \\ T(x)+T(y) &= 2x+2y \end{aligned}\right\} \Rightarrow T(x+y) = T(x)+T(y) \quad \forall x,y \in \mathbb{R}.$$

(ii)
$$\left.\begin{aligned} T(c\cdot x) &= 2cx \\ cT(x) &= c\cdot(2x) = 2cx \end{aligned}\right\} \Rightarrow T(cx) = cT(x) \quad \forall c \in \mathbb{R},\ x \in \mathbb{R}.$$

By (i), (ii), T is linear.

12). $T : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2x+3.$

Method 1. Check additive property:

(i) $T(x+y) = 2(x+y) + 3 = 2x + 2y + 3$

$T(x) + T(y) = (2x+3) + (2y+3) = 2x + 2y + 6$ } they are not equal.

So T doesn't respect addition, therefore T is not linear.

Method 2. $T(0) = 2 \cdot 0 + 3 = 3 \neq 0$, so T is not linear by Prop. 0.

(3) $T: \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto x^2$ Ex: Show that T also violates (ii).

What about Prop. 0? $T(0) = 0^2 = 0$. So we can't quickly conclude that T is not linear.

Additive property?

$$\left.\begin{aligned} T(x+y) &= (x+y)^2 = x^2 + y^2 + 2xy \\ T(x) + T(y) &= x^2 + y^2 \end{aligned}\right\}$$

not ~~always~~ equal. eg. when $x = y = 1$,

So T doesn't respect addition, therefore it's not linear.

Another way to write the soln: (Use a counter-example.) For $x = y = 1$,

we have $T(x+y) = T(1+1) = T(2) = 4$ while $T(x) + T(y) = x^2 + y^2$

$= 1^2 + 1^2 = 2$, so $T(x+y) \neq T(x) + T(y)$ and T is not linear.

(f) $T : \mathbb{R} \to \mathbb{R}^2, \quad x \mapsto \begin{bmatrix} 2x \\ x^2 \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$ $\begin{array}{l} \text{linear} \\ \text{not linear} \end{array}$

Soln:

$$T(x+y) = \begin{bmatrix} 2(x+y) \\ (x+y)^2 \end{bmatrix} = \begin{bmatrix} 2x+2y \\ x^2+y^2+2xy \end{bmatrix} \checkmark$$

not equal

$$T(x)+T(y) = \begin{bmatrix} 2x \\ x^2 \end{bmatrix} + \begin{bmatrix} 2y \\ y^2 \end{bmatrix} = \begin{bmatrix} 2x+2y \\ x^2+y^2 \end{bmatrix} \checkmark$$

Since $\underbrace{(x+y)^2 \neq x^2+y^2 \text{ in general}}_{T_2(x) \text{ is not linear}}$, $T(x+y) \neq T(x)+T(y)$ in general.

Therefore T is not linear.

Intuition: "exponents" are bad.

(5). $T: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} 3x-y \\ y+z \\ 2x-z \end{bmatrix}$.

Additive property: Ex. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = T\left(\begin{bmatrix} x+x' \\ y+y' \\ \cdots \end{bmatrix}\right) = \begin{bmatrix} 3(x+x') - (y+y') \\ \cdots \\ \cdots \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = \begin{bmatrix} 3x-y \\ \vdots \end{bmatrix} + \begin{bmatrix} 3x'-y' \\ \vdots \end{bmatrix} = \begin{bmatrix} 3x-y+3x'-y' \\ \vdots \end{bmatrix}$$

Scaling: $T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} 3cx - cy \\ cy + cz \\ 2cx - cz \end{bmatrix}$)).

$$cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} 3x-y \\ y+z \\ 2x-z \end{bmatrix} = \begin{bmatrix} 3cx - cy \\ cy + cz \\ 2cx - cz \end{bmatrix}$$

Since T respects add. and scalar mult, T is linear.

Ex: The map $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2x \\ 3x-y \end{bmatrix}$ is linear,

while the map $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \sin x \\ 2y \\ 2\pi xy \end{bmatrix}$ is not.

Note: Every map $T : \mathbb{R}^n \to \mathbb{R}^m$ must send an input $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to an output of the form $\begin{bmatrix} T_1(x_1, \ldots, x_n) \\ \vdots \\ T_m(x_1, \ldots, x_n) \end{bmatrix}$ $\to$ $T_1, T_2, \ldots, T_m$ are the "component maps" and together they are equivalent to T.

Note: T is linear $\overset{*}{\Longleftrightarrow}$ $T_1, T_2, \ldots, T_m$ all take linear comb. of the entries in the input vector.

Pf of (*) :

(1) T respects $+$ $\iff$ $T(v+w) = T(v) + T(w) \quad \forall v, w \in \mathbb{R}^n$

$$\iff \begin{bmatrix} T_1(v+w) \\ \vdots \\ T_m(v+w) \end{bmatrix} = \begin{bmatrix} T_1(v) \\ \vdots \\ T_m(v) \end{bmatrix} + \begin{bmatrix} T_1(w) \\ \vdots \\ T_m(w) \end{bmatrix} \quad \forall v, w \in \mathbb{R}^n$$

$$\iff T_i(v+w) = T_i(v) + T_i(w) \quad \forall v, w \in \mathbb{R}^n \ \forall 1 \le i \le m$$

$\iff$ T_i respects $+$ for all $1 \le i \le m$

(2) Similarly, T respects scaling $\iff$ T_i respects scaling $\forall i$.

(1) + (2) $\Rightarrow$ T is linear iff $T_1, \ldots, T_m$ are all linear.

Next time: formulas of lin. maps ; "linear maps $\equiv$ matrix mult."