Math 2135. Linear Algebra

Course Information.

Instructor: Tianyuan Xu (Eddy)

Website: https://math.colorado.edu/~tixu6187/2135.html

- has Canvas link
- lecture notes & HW posted under "LECTURES" tab

Office Hours: Zoom appointments

Grading: HW 20%, Midterm 20% × 2, Final 40%

HW: — to be submitted on Canvas/Assignments, pdf only

— due on Wednesday nights 11:59 pm.

(the deadline is strict ; no late submission possible)

— HW1 due on Sept 1.

— posted the previous Wed. → Aug. 25.

Textbook: " Linear Algebra and its Applications, Fifth Edition "

by Lay, Lay and McDonald.

— available in Canvas/Files.

On to the math.

Today. Systems of linear equations. Matrices. Row operations.

1. Systems of linear equations

(L.E.S.)

Def. By a system of linear equations we mean a finite set of equations of the form

$$(*)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} is a constant $\forall\, 1 \le i \le m,\ 1 \le j \le n$ and $b_1, \ldots, b_m$ are constants.

Def: (1) (Solutions) A <u>solution</u> of a linear system of the form $*$ is just a tuple of values $(x_1, x_2, \dots, x_n)$ for which all the equations hold.

(2). (Consistency) We say a linear equation system is <u>consistence</u> if it has at least one soln and <u>inconsistent</u> otherwise.

e.g. $\begin{cases} x+y=3 \\ 2x+2y=5 \end{cases}$ is inconsistent ; $\begin{cases} x+y=3 \\ y=1 \end{cases}$ is consistent with exactly one soln $(2,1)$;

$\{ x-y=1$ is consistent and has an infinite soln set $\{(t+1, t) : t \text{ is a constant}\}$

(3) (Equivalence of SELs) Two linear equation systems are <u>equivalent</u> if they have the same soln set. **eg.** $\begin{cases} x+y=3 \\ y=1 \end{cases} \Leftrightarrow \begin{cases} x=2 \\ y=1 \end{cases}$

Central question: Given a linear equation (system $*$), how can we tell if it's consistent? If it is, how can we find all its solns? In particular, how many solutions are there?

Already a nontrivial fact: (Trichotomy) A linear equation system always has 0, 1, or infinitely many solns.

We'll define matrices and address the central questions using matrices.

2. Augmented matrices and row operations.

(a). Encode: to solve a linear system, we'll first encode it with a matrix

$$(*).\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

eg $\begin{cases} 2x + y = 3 \\ y - x = 4 \end{cases}$

$\Updownarrow$

$$\begin{array}{c} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix} \\ \begin{array}{ccc} x & y & \end{array} \end{array}$$

by the matrix. (a rectangular array)

the "coefficient matrix"

the "augmented matrix" $\leftarrow A_* = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

$\quad\quad x_1 \quad x_2 \quad\quad x_n$

rows $\leftrightarrow$ equations

cols $\leftrightarrow$ variables

Conversely, we can (and will) recover the system from its matrix.

(b). **Manipulate:** Recall that we can solve linear equation systems by eliminating variables. We'll translate the method to matrices, paying attention to the types of matrix operations corresponding to the steps.

Eg. 1.

$$\begin{cases} x - 3y = 1 & ① \\ 2x = 4 & ② \end{cases} \qquad \begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$① \leftrightarrow ②$ $\downarrow$ $\qquad$ $\downarrow$ interchange two rows

$$\begin{cases} 2x = 4 & ①' \\ x - 3y = 1 & ②' \end{cases} \qquad \begin{bmatrix} 2 & 0 & 4 \\ 1 & -3 & 1 \end{bmatrix}$$

$①' \times \frac{1}{2}$ $\downarrow$ $\qquad$ $\downarrow$ scaling (every entry in) a row by a nonzero constant.

$$\begin{cases} x = 2 & ①'' \\ x - 3y = 1 & ②'' \end{cases} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

$$\begin{cases} x = 2 & ①'' \\ x - 3y = 1 & ②'' \end{cases}$$

"eliminate x from ②''"

$$\begin{cases} x = 2 & ①''' \\ -3y = -1 & ②''' = ②'' - ①'' \end{cases}$$

scale an equation

$$\begin{cases} x = 2 \\ y = \frac{1}{3} \end{cases}$$

DONE.

$$\begin{array}{c} R1 \\ R2 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} \rightarrow \text{nice} : x = 2$$

subtract a multiple of a row (Row 1) from another row (Row 2)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$

scale a row by a nonzero constant

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \rightarrow \text{nice} : y = \frac{1}{3}.$$

E.g. 2.

$$\begin{cases} 2x - 3y = 1 & ① \\ 4x + y = 23 & ② \end{cases}$$

$$\Big\downarrow ① \times 2$$

$$\begin{cases} 4x - 6y = 2 & "①" \\ 4x + y = 23 & "②" \end{cases}$$

$$\Big\downarrow ② - ①$$

$$\begin{cases} 4x - 6y = 2 & ① \\ 7y = 21 & ② \end{cases}$$

$$\Big\downarrow ② \times \tfrac{1}{7}$$

$$\begin{cases} 4x - 6y = 2 & ① \\ y = 3 & ② \end{cases}$$

Legal operations.

- $R_i \Leftrightarrow R_j$
- $R_i \to c \cdot R_i, c \neq 0$
- $R_i \to R_i + c \cdot R_j$.

$$\begin{array}{cc} R1 & \\ R2 & \end{array} \begin{bmatrix} 2 & -3 & 1 \\ 4 & 1 & 23 \end{bmatrix}$$

"scale"

$$\begin{bmatrix} 4 & -6 & 2 \\ 4 & 1 & 23 \end{bmatrix}$$

can be combined

$\boxed{R_2 - 2 \cdot R_1}$

replace R_2 with $R_2 - R_1$

$$\begin{bmatrix} 4 & -6 & 2 \\ 0 & 7 & 21 \end{bmatrix}$$

"scale"

$$\begin{bmatrix} 4 & -6 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} 4x - 6y = 2 & ① \\ y = 3 & ② \end{cases}$$

$$\Big\downarrow \ ① + ② \times 6$$

$$\begin{cases} 4x = 20 & ① \\ y = 3 & ② \end{cases}$$

$$\Big\downarrow \ ① \times \tfrac{1}{4}$$

$$\begin{cases} x = 5 \\ y = 3 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} 4 & -6 & 2 \\ 0 & 1 & 3 \end{bmatrix} \end{array} \begin{array}{l} R1 \\ R2 \end{array}$$

"replacement": $R_i \to R_i + c \cdot R_j$

$$\begin{bmatrix} 4 & 0 & 20 \\ 0 & 1 & 3 \end{bmatrix} \begin{array}{l} R1 + 6 \cdot R_2 \\ R2 \end{array}$$

scale

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{cases} x = 5 \\ y = 3. \end{cases}$$

Eg. 3.

$$\begin{cases} \quad\quad\quad 3z = 9 \\ 2x \quad\quad - z = 5 \\ \quad\quad 2y + z = 1 \end{cases}$$

Ex. Try to solve the above system and write down the corresponding operations on matrices.

Note: We've observed that the following types of row operations do not change the soln set of the encoded S.E.L.

(E1) Interchange : interchang two rows

(E2) Scaling : multiply a row by a nonzero scalar

(E3) Replacement : add a multiple of a row to another : $R_i \leftarrow R_i + cR_j$.

Fact: Using the three operations E1, E2, E3, we may transform the aug. matrix of any S.E.L to a "nice" form that makes it easy to find the solns of the SEL.

Next time: explain how to do this after defining what "nice" means.