# MATH 2130. REVIEW FOR CHAPTERS 5 AND 6

## Final Exam Information:

The final exam will be available on Canvas in Assignments/Final Exam from 11:59 am to 11:59 pm of Wednesday, May 5, 2021. The exam is designed so that you should be able to finish it in two hours, but to be flexible I'm giving you the twelve-hour window, and you are allowed to take as much time as you need to finish the problems, as long as you finish the submission by 11:59 pm. Given the long twelve-hour window, late submission will absolutely not be accepted.

The final exam will be cumulative, despite the fact that this review sheet only concerns Chapters 5 and 6 of the book, which are the parts covered after the second midterm. For review of Chapters 1–4, please see the two previous review sheets.

## Review Problems:

1. Read the sections 5.1–5.4 and 6.1–6.4 in the book. For Section 6.4 you can stop right after the statement of Theorem 11.

2. Let $A$ be an $n \times n$ matrix.

   (a) State the definition of an eigenvector and the corresponding eigenvalue for $A$. Can an eigenvector be the zero vector? At most how many eigenvalues can $A$ have?

   (b) Given a vector $v$, how should we check whether $v$ is an eigenvector of $A$? Given a scalar $\lambda$, how should we check whether $\lambda$ is an eigenvalue of $A$? Do exercises 5 and 7 in Section 5.1.

   (c) What is the characteristic polynomial of $A$? How does it relate to the problem of finding eigenvalues of $A$, and what is the algebraic multiplicity of each eigenvalue? Do Exercise 15 of Section 5.2.

   (d) For each eigenvalue $\lambda$ of $A$, how many eigenvectors does the corresponding eigenspace

$E_\lambda$ (i.e., the set of eigenvectors of $A$ with eigenvalue $\lambda$) have? How can we find a basis of $E_\lambda$?

(e) Recall that for each eigenvalue $\lambda$ of $A$, the dimension of the eigenspace $E_\lambda$ is called the geometric multiplicity of $A$. What can we say about the relationship between the geometric multiplicity and the algebraic multiplicity of each eigenvalue?

(f) What does it mean for $A$ to be diagonalizable? How does diagonalizability relate the algebraic and geometric multiplicities of the eigenvalues of $A$? Read examples 3, 4 and 6 in Section 5.3.

3. Let $A$ be an $n \times n$ matrix?

   (a) What does it mean to diagonalize $A$? Describe how to test whether it can be done, and how to do it when possible.

   (b) Do exercises 1, 3, and 11 in Section 5.3.

4. Let $V$ and $W$ be abstract vector spaces. Let $B = \{b_1, b_2, \cdots, b_n\}$ be a basis of $B$, and let $C = \{c_1, c_2, \cdots, c_m\}$ be a basis of $C$.

   (a) Given a vector $w \in W$, what is the coordinate vector $[w]_C$ of $w$ with respect to $C$? Which vector space is $[w]_C$ in?

   (b) Find $[w]_C$ for each of the following cases.

      i. $W = \mathbb{R}^3$, $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $C$ is the standard basis

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

ii. $W = \mathbb{R}^3$, $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $C$ is the basis

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

iii. $W = P_2$, the set of polynomials with real coefficient of degree at most 2, $w = 2t + 3t^2$, and $C$ is the basis $\{t^2, t, 1\}$.

iv. $W = P_2$, the set of polynomials with real coefficient of degree at most 2, $w = 2t + 3t^2$, and $C$ is the basis $\{2t, t^2 - 2, 2\}$.

(c) Now let $T : V \to W$ be a linear map. How do we find the matrix $[T]_B^C$, the matrix of $T$ relative to $B$ and $C$? In particular, how do we find the $j$-th column of $[T]_B^C$ for $1 \leq j \leq n$? Do exercises 1, 2 in Section 5.4.

(d) Find $[T]_B^C$ in the following cases.

i. $V = P_2$, $B = \{1, t, t^2\}$, $W = P_2$, $T : V \to W$ is the "differentiation" map, and $C$ is as in (b.iii).

ii. $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, $T : V \to W$ is the "differentiation" map, and $C$ is as in (b.iii).

iii. $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, $T : V \to W$ is the "differentiation" map, and $C$ is as in (b.iv).

iv. $V = P_3$, $B = \{1, t, t^2, t^3\}$, $W = P_2$, $T : V \to W$ is the "differentiation" map, and $C$ is as in (b.v).

5. Let $A$ be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by $T(x) = Ax$ for all $x \in \mathbb{R}^n$. How does the diagonalizability of $A$ relate to the problem of trying to find a basis $B$ of $\mathbb{R}^n$ for which $[T]_B^B$ is diagonal? Read Example 3 and do exercises 13 and 15 in Section 5.4.

6. Let $B = \{b_1, b_2, \cdots, b_n\}$ and $C = \{c_1, c_2, \cdots, c_n\}$ be two bases of $R^n$.

   (a) How do we find the change-of-basis matrix $\mathcal{P}_{C \leftarrow B}$ from $B$ to $C$? What equation should hold for $\mathcal{P}_{C \leftarrow B}$ and the coordinate vectors $[v]_B, [v]_C$ where $v$ is an arbitrary vector in $\mathbb{R}^n$.

   (b) How are the two change-of-basis matrices $\mathcal{P}_{C \leftarrow B}$ and $\mathcal{P}_{B \leftarrow C}$ related?

7. Let $u, v$ be vectors in $\mathbb{R}^n$.

   (a) State the definition of the inner product $u \cdot v$ of $u$ and $v$.

   (b) Explain why $u \cdot u \geq 0$ for all $u \in \mathbb{R}^n$.

   (c) Prove that for each $u \in \mathbb{R}^n$, the map "dot with $u$", i.e., the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(v) = u \cdot v$, is a linear map.

   (d) Define the length of $u$. Explain how to "normalize" any given vector $u$, i.e., how to find a unit vector with length 1 which points in the same direction as $u$. Do Exercise 9 in Section 6.1.

   (e) How is the distance between $u$ and $v$ defined? Do Exercise 13 in Section 6.1.

   (f) Explain, in terms of inner products, what it means for $u$ and $v$ to be orthogonal.

8. Let $B = \{u_1, u_2, \cdots, u_n\}$ be an orthogonal basis of $\mathbb{R}^n$ and let $y \in \mathbb{R}_n$.

   (a) Let $1 \leq j \leq n$. What's the $j$-th coordinate in the vector $[y]_B$?

   (b) Write down the formula for the orthogonal projection $\hat{y}_j$ of $y$ onto $u_j$.

   (c) Let $1 \leq i < j \leq n$. Are the orthogonal projections $\hat{y}_i$ and $\hat{y}_j$ orthogonal? Why?

   (d) Let $I$ be a subset of $B$ and let $W$ be the subspace of $\mathbb{R}^n$ spanned by the elements in $I$. How do we find the orthogonal projection $\text{proj}_W y$ of $y$ onto $W$, and how do we find the distance from $y$ to $W$? Read examples 3, 4 and do problems 11, 13, 15 in Section 6.3.

9. Explain what the Gram–Schmidt algorithm does and how it runs. Read examples 1,2 and do exercises 1, 2, 3 in Section 6.4.