Math 2130. Review for Midterm II

Midterm Information:

The midterm will be available on Canvas in Assignments/Midterms from 5:59 pm to 11:59 pm of Monday, April 5, 2021. Please upload your completed midterm in the same way you submit your homework by 11:59 pm. The exam is designed so that you should be able to finish it in one hour, but to be flexible I'm giving you the six-hour window, and you are allowed to take as much time as you need to finish the problems, as long as you finish the submission by 11:59 pm. Given the long six-hour window, late submission will absolutely not be accepted.

Review Problems:

1. Read the following sections in the book: 2.1–2.3, 2.8–2.9, 3.1–3.3, 4.1–4.7. For Section 3.3 you only need to read the part about "Determinant as Area or Volume".

2. Let A, B be $n \times n$ matrices.

   (a) State the definition of A^{-1}.

   (b) Find the inverse of the matrix

   $$C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

   (c) Recall that $\det(AB) = \det A \det B$. Use this fact (which you should know) and Part (a) to prove that $\det(A^{-1}) = 1/\det A$.

   (d) Recall that $\det A = \det A^T$, where A^T stands for the transpose of A. Use this fact (which you should know) and the previous parts to compute $\det(C^2(C^T)^3C^{-1})$ for the matrix C from Part (b).

3. Let A be an $n \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n$ the map defined by $T(x) = Ax$ for any $x \in \mathbb{R}^n$. Fill in the blanks below for the characterization of invertible matrices.

Theorem 1. *The following are equivalent.*

(a) *A is invertible.*

(b) *An echelon form of A contains ... pivot columns.*

(c) *An echelon form of A contains ... zero rows.*

(d) *The columns of A are ....*

(e) *The columns of A span ....*

(f) *The rank of A is ....*

(g) *The nullity of A is ....*

(h) *The map T is injective.*

(i) *The map T is ....*

(j) *The map T is bijective.*

(k) *The kernel of T is ....*

(l) *The image of T is ....*

(m) *The equation $Ax = 0$ has ... solution.*

(n) *The equation $Ax = b$ has ... solution for all $b \in \mathbb{R}^n$.*

(o) $\det A$ *... zero.*

(p) *A^T is ....*

4. (a) State the definition of the null space, column space, and row space of a matrix.

   (b) Let A be an 4×7 matrix with 4 pivot columns. Find the dimensions of the null space, column space and row space of A, then do the same for the transpose A^T. Is $\operatorname{Col} A = \mathbb{R}^4$? Is $\operatorname{Null} A = \mathbb{R}^3$?

(c) Let

$$B = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Find bases for $\operatorname{Null} B, \operatorname{Col} B, \operatorname{Row} B$, then conclude what their dimensions are.

5. Let

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

(a) Compute $\det A$, then use it to compute $\det\left(\frac{1}{2}A\right)$.

(b) Let B be the matrix obtained from A by interchanging the top and bottom row. What is $\det B$ and why?

(c) Let C be the matrix obtained from B by adding 3 times the first row to the third row. What is $\det C$ and why?

6. Let V, W be vector spaces and let $T : V \to W$ be a linear map.

(a) State the definition of a subspace of V.

(b) Prove that the image of T is a subspace of W.

(c) Let w be a nonzero element in W. Prove that the set $\{v \in V : T(v) = w\}$ is not a subspace of V.

7. Let P_3 be the vector space of all polynomials in indeterminate t of degree at most 3.

(a) Prove that the differentiation map $d : P_3 \to P_3$, i.e., the map defined by $d(f(t)) = f'(t)$ for any $f(t) \in P_3$, is a linear map. Clearly point out any facts from calculus you are using.

(b) Describe the kernel of d, then find a basis of the kernel.

(c) Describe the image of d, then find a basis of the image.

(d) Verify that the set $B = \{1, 2+t, 3+t+t^2, 4+t^3\}$ is a basis of P_3.

(e) Let $g(t) = t^3 + 2t^2 + t - 10$ and let B be as in Part (d). Find $[g(t)]_B$.

8. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ where

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

(a) Prove that both B and C are bases of $\mathbb{R}^2$.

(b) Find $\mathcal{P}_{C \leftarrow B}$, the change-of-basis matrix from B to C.

(c) Use the previous part to find $\mathcal{P}_{B \leftarrow C}$.

(d) Suppose $[v]_C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for some $v \in \mathbb{R}^2$. Find v and $[v]_B$.