Math 2130. Review for Midterm I

Midterm Information:

The midterm will be available on Canvas in Assignments/Midterms from 5:59 pm to 11:59 pm of Monday, March 1, 2021. Please upload your completed midterm in the same way you submit your homework by 11:59 pm. The exam is designed so that you should be able to finish it in one hour, but to be flexible I'm giving you the six-hour window, and you are allowed to take as much time as you need to finish the problems, as long as you finish the submission by 11:59 pm. Given the long six-hour window, late submission will absolutely not be accepted.

Review Problems:

1. Read Sections 1.1–1.5, 1.7–1.9, and 2.1–2.2 of the textbook. For Section 2.2 you can ignore the material after Example 4.

2. State the definition of linear maps, i.e., explain what it means for a map $T : V \to W$ between vector spaces V, W to be linear. Make sure you use the proper quantifiers like "for all" or "for some" where necessary.

3. Consider the following four matrices.

$$A = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 6 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) Determine which of the four matrices is/are in echelon form. For each matrix not in echelon form, point out one reason why it isn't.

   (b) Which of the matrices is/are in reduced echelon form? Explain.

(c) Out of the four matrices, pick any that is not in the reduced echelon form, then compute its reduced echelon form and find the pivot column(s) of the matrix.

4. Consider the following system of linear equations.

$$\begin{cases} -2x + 5y + 5z - 5w &= 3 \\ x - 4y - z &= 1 \\ x - 2y &= 0 \end{cases}$$

(a) Write the system as a vector equation.

(b) Explain, without doing any computation, why the system must have infinitely many solutions.

5. Consider the matrix equation

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & 4 & 2 \\ 2 & -14 & 1 & 10 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ -3 \\ 7 \\ 7 \end{bmatrix}.$$

Solve the equation. Write the general solution in parametric vector form, then describe the solution set geometrically.

6. Determine whether each of the following maps is linear. Explain your reasoning.

$$\begin{aligned} T_1 &: \mathbb{R} \to \mathbb{R}, & T_1(x) &= 2x + 3 \\ T_2 &: \mathbb{R}^2 \to \mathbb{R}^2, & T_2(x, y) &= (x + y, y - 1) \\ T_3 &: \mathbb{R}^3 \to \mathbb{R}^3, & T_3(x, y, z) &= (x^2, y + 2z, x + y + z) \\ T_4 &: \mathbb{R}^4 \to \mathbb{R}^4, & T_4(x, y, z, w) &= (x + y, x, y + 2z, z - x - y) \end{aligned}$$

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which first reflects points through the x-axis and

then reflects points through the line $y = x$.

(a) Find the matrix of T with respect to the standard basis.

(b) Determine if T is surjective.

(c) Determine if T is injective.

8. Consider the matrix

$$A = \begin{bmatrix} 3 & 5 & h \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(a) Find all value(s) of h for which the third column of A is in the span of the first two columns of A.

(b) Find all value(s) of h for which the columns of A are linearly independent.

(c) Find all value(s) of h for which the columns of A span $\mathbb{R}^3$.

(d) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$. For which values of h is T surjective, and for which values of h is T injective? You may use the results from the previous parts.

9. (a) **[1pt]** Suppose A, B are matrices such that A is 5×3, AB is defined, and AB is 5×7. What is the size of B?

(b) **[4pts]** Let

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

Determine which of the products AB, BA, A^TB^T, B^TA^T are defined, and compute all the products that are.

10. Find two 2×2 matrices A, B such that $AB \neq BA$.

11. Find three 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$.

12. Recall that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc \neq 0$ is invertible and has inverse

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Use this fact to solve the vector equation

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

(*Hint*: Rewrite the vector equation as a matrix equation first.)