Math 2130. Lecture 9.

02.05.2021

Last time:

- geometry of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
- soln sets of homogeneous and non-homogeneous systems.

$C\vec{x} = \vec{0}$ $\qquad\qquad$ $C\vec{x} = \vec{b}$ $(\vec{b} \neq 0)$

Done with Sections 1.1 – 1.7. (We'll skip 1.6.)

Today.

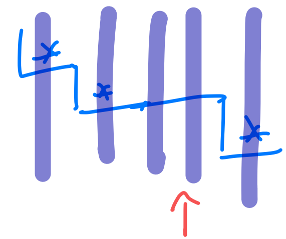
- Review / Loose ends: numerical restrictions on spanning / linearly ind. sets in $\mathbb{R}^n$.
- Introduction to linear transformations.

1. Size of spanning / lin. ind. sets.

key: # pivots $\leq$ min (# cols of C, # rows of C).

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$. Let $C = [\vec{v}_1 | \cdots | \vec{v}_k]$.

Recall that S spans $\mathbb{R}^n$ iff $EF(C)$ has no zero row, iff every row in C is pivot.



It follows that if S spans $\mathbb{R}^n$, then

n = # rows in C = # pivot row = # pivots in $EF(C) \leq$ # cols in $C = k$

i.e., a spanning set in $\mathbb{R}^n$ must have at least n vectors.

e.g. for a set S to span $\mathbb{R}^2$, S needs to have at least two vectors.
(span of one vector is at most a line, not a plane.)

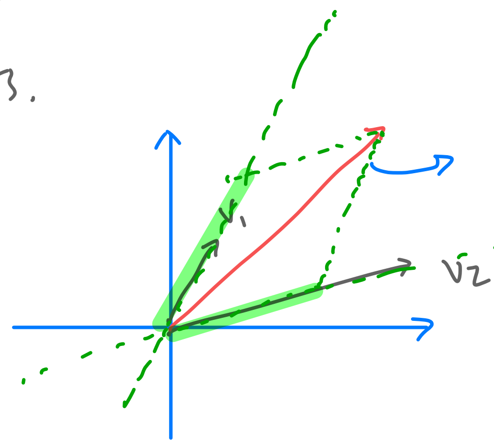
Similarly, recall that S is lin. ind iff every col in C is pivot.

It follows that if $S \subseteq \mathbb{R}^n$ is lin ind, then

$$k = \# \text{ cols in } C = \# \text{ pivot cols} = \# \text{ pivots in } C \le \# \text{ rows in } C = n,$$

i.e., a lin ind. set in $\mathbb{R}^n$ can have at most n elts.

e.g. for a set S to be lin. ind in $\mathbb{R}^2$, S can have at most two vectors.



a lin comb. of v_1 and v_2,

so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not lin. ind.

Note: The converses of the two facts don't hold.

True: if $|S| < n$, then S doesn't span $\mathbb{R}^n$.

Converse: If $|S| \geq n$, then S spans $\mathbb{R}^n$. $\rightarrow$ not true.

e.g. $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

True: If $|S| > n$, then S is not lin ind in $\mathbb{R}^n$.

Converse: if $|S| \leq n$, then S is lin ind in $\mathbb{R}^n$. $\rightarrow$ not true

e.g. $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$.

Examples. Determine if S spans $\mathbb{R}^n$ and if S is lin. ind.

(1) $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \in \mathbb{R}^2$.

Soln: $|S| = 3 > 2$, so S is not lin ind.

Spanning: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}}_{\text{E.F.}}$ There's no zero row in the E.F. so S spans $\mathbb{R}^2$.

(2) $T = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \in \mathbb{R}^3$

Soln: $|T| = 2 < 3$, so T does not span $\mathbb{R}^3$; T contains $\vec{0}$, so T is not lin. ind.

(3). $U = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$.

Soln: HW.

2. Linear transformations.

Def: A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear transformation</u> if it respects vector addition and scalar multiplication, i.e., if

"linearity"

$$\begin{cases} T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) & \forall \vec{v}, \vec{w} \in \mathbb{R}^n \\ \text{and} \quad T(c \cdot \vec{v}) = c \cdot T(\vec{v}) & \forall c \in \mathbb{R}, \vec{v} \in \mathbb{R}^n. \end{cases}$$

Note: The above def. can be generalized to so-called maps between "<u>vector spaces</u>", which are certain sets where addition and scalar multiplication make sense and behave well.

the elts are called "vectors"

e.g. $P_3 = \{\text{polynomials in } \mathbb{R}[t] \text{ of degree} \leq 3\}$

$P_2 = \{ \text{- - - - - - - - - - -} \leq 2\}$

formal def

$T: P_3 \to P_2 \quad f(t) \mapsto f'(t)$ e.g. $t^2 \mapsto 2t$ is linear!

Note: The map $T: P_3 \to P_2$ given by formal diff is linear because diff respects add. and scalar mult. by calculus.

$$(f+g)' = f'+g', \qquad (c\cdot f)' = c\cdot f'.$$

Using linearity. Let $S=\{\vec{v}_1,\dots,\vec{v}_k\}\subseteq \mathbb{R}^n$ be a spanning set of $\mathbb{R}^n$. Suppose we have a lin. transf. $T:\mathbb{R}^n\to\mathbb{R}^m$ and suppose we know $\underbrace{T(\vec{v}_1),\ T(\vec{v}_2),\ \dots,\ T(\vec{v}_k)}_{\text{known}}$. Then in theory we know $T(\vec{v})$ for all $\vec{v}\in\mathbb{R}^n$.

Reason: S spans $\mathbb{R}^n$, so $\vec{v} = c_1\vec{v}_1+\dots+c_k\vec{v}_k$ for some $c_1,\dots,c_k$. Linearity of T then forces

$$\underbrace{T(\vec{v})}_{\text{unknown}} = T(c_1\vec{v}_1+\dots+c_k\vec{v}_k)$$
$$= T(c_1\vec{v}_1)+T(c_2\vec{v}_2)+\dots+T(c_k\vec{v}_k)$$
$$= c_1 T(\vec{v}_1)+c_2T(\vec{v}_2)+\dots+c_kT(v_k)$$

Examples

(1) Say we have a lin. map $T: \mathbb{R}^2 \to \mathbb{R}^3$ with

$$\underbrace{T\left(\overset{\vec{v}_1}{\begin{bmatrix}1\\2\end{bmatrix}}\right)}_{\text{known}} = \begin{bmatrix}2\\1\\3\end{bmatrix}, \qquad \underbrace{T\left(\overset{\vec{v}_2}{\begin{bmatrix}2\\3\end{bmatrix}}\right)}_{\text{known}} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then

$$T\left(\begin{bmatrix}3\\5\end{bmatrix}\right) = T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\8\end{bmatrix}\right) = T(\vec{v}_1 + 2\vec{v}_2) = T(\vec{v}_1) + 2T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} + 2\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = T(\vec{v}_1 - \vec{v}_2) \boxed{= T(\vec{v}_1 + (-1)\cdot\vec{v}_2) = T(\vec{v}) + (-1)T(\vec{v}_2)}$$

$$= T(\vec{v}_1) - T(\vec{v}_2) = \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

(2). Say we have a lin map $\mathcal{U}: \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\mathcal{U}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathcal{U}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathcal{U}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

Then for an arbitrary elt $\vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$, since

$$\vec{v} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix},$$

we have

$$\mathcal{U}(\vec{v}) = \mathcal{U}\left(a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = a\,\mathcal{U}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + b\,\mathcal{U}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + c\,\mathcal{U}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}a-c\\2a+b-c\\3a-c\end{bmatrix}.$$

More on linear maps next time.

□