Math 2130. Lecture 8.

02.03.2021.

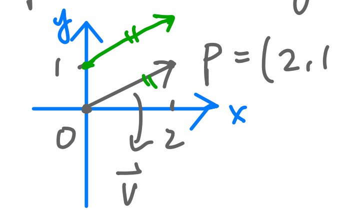
Last time: linear independence:

- def. $\left(x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{0} \Rightarrow x_1 = x_2 = \cdots = x_k = 0 \right)$
- echelon form criterion ($EF\left(\left[\vec{v}_1 \middle| \cdots \middle| \vec{v}_k\right]\right)$ has a pivot in every col.)
- other special criteria: for small sets, or when a nontrivial zero lin comb. is easy to find

Today: · geometry of vectors · homogeneous vs. non-homogeneous LES.

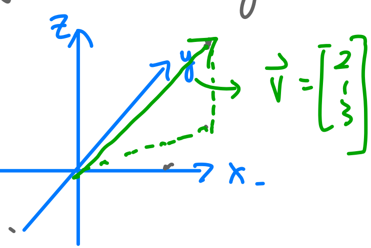
1. Geometry of Vectors.

- We identify $\mathbb{R}^2$ with the 2-D plane and identify each vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ with the arrow $\overrightarrow{OP}$ from the origin O to the point $P = (a, b)$. e.g. $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftrightarrow$



note the comma in the tuple notation

- We identify $\mathbb{R}^3$ with the 3-D space and identify each vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ with the arrow $\overrightarrow{OP}$ from the origin O to the point $P = (a, b, c)$. e.g. $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \leftrightarrow$

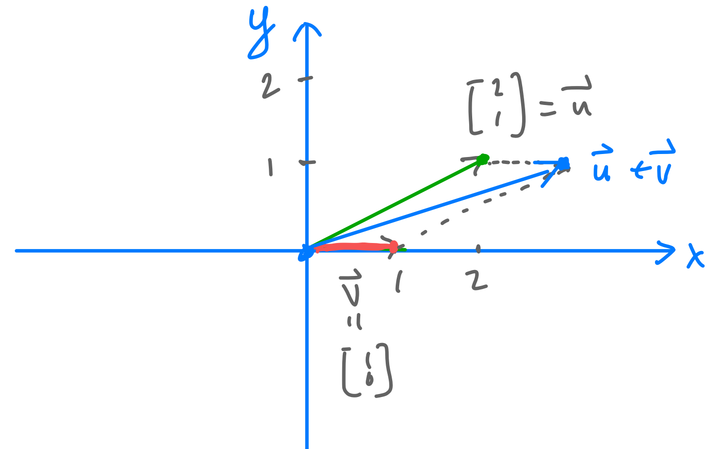


- We also identify arrows with the same direction and length as equal vectors.

· (Parallelogram law) For $\vec{u}, \vec{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$, $\vec{u}+\vec{v}$ corresponds to the (arrow from the origin to) the fourth vertex of the parallelogram (▱) whose other vertices are 0, $\vec{u}$, and $\vec{v}$.

e.g.

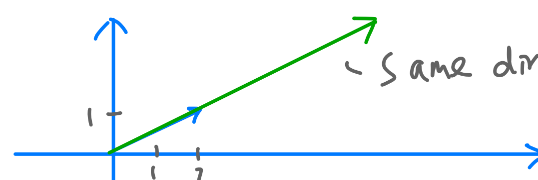
$$\underset{\vec{u}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + \underset{\vec{v}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \longleftrightarrow$$



· Scaling a vector scales the length without changing its direction.

e.g.

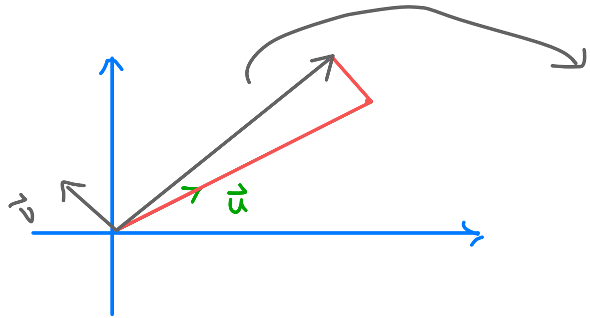
$$3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}. \longleftrightarrow$$



← Same direction, but three times the length.

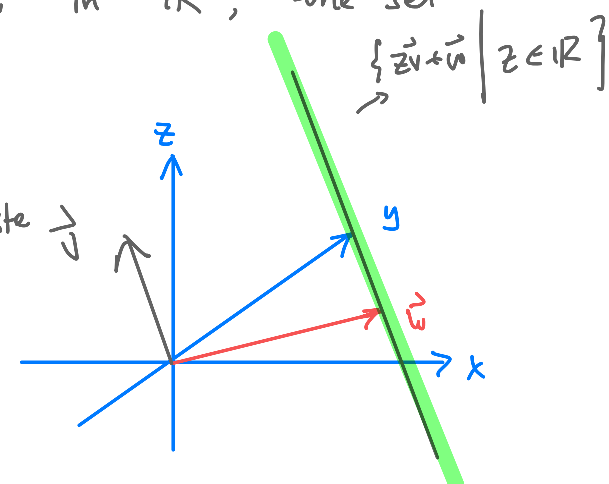
· (Linear combinations) Now that we can add and scale vectors, we can draw their linear comb. as well.

e.g.



$3\vec{u}+\vec{v}$?

E.g. Given a vector two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, the set $\{z\vec{v}+\vec{w} \mid z\in\mathbb{R}\}$ can be described as the line in $\mathbb{R}^3$ that passes through $\vec{w}$ in/opposite $\vec{v}$ the direction of $\vec{v}$.



More drawings in homework...

2. Homogeneous vs. Non-homogeneous systems

Def. An LES of the form (*) $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$ is called

homogeneous if $b_1 = b_2 = \cdots = b_m = 0$. Similarly, we say a vec eq.

$x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{v}$ is homogeneous if $\vec{v} = \vec{0}$ and say a matrix eq. ($A\vec{x} = \vec{v}$)

is homogeneous if $\vec{v} = \vec{0}$. (So, a LES is hom. iff its vec eq. is hom., iff its matrix eq. is hom.)

Note: $\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a soln of (*) iff (*) is homogeneous.

E.g. $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\uparrow C$ (left block), $\uparrow \vec{b}$ (last column); R.E.F.

$\boxed{\vec{v} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \text{ is a soln of } C\vec{x} = \vec{b}.}$

(1) The eq. $C\vec{x} = \vec{b}$ is not homogeneous.

Recall its soln set is $\left\{ z \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} : z \in \mathbb{R} \right\}$, i.e., it's the line in $\mathbb{R}^3$ that passes the point $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ in the direction of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(2). The eq. $C\vec{x} = \vec{0}$ is homogeneous. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 0 \\ 9 & 10 & 11 & 0 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

It has soln set $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - z = 0 \\ y + 2z = 0 \\ z \in \mathbb{R} \end{matrix} \right\} = \left\{ \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$,

i.e., it's the line in $\mathbb{R}^3$ that goes through the origin in the direction $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Thm. Consider a hom. matrix equation $C\vec{x} = \vec{0}$ and a non-hom eq. $C\vec{x} = \vec{b}$ $(\vec{b} \neq 0)$

Suppose that both equations are consistent. Let $S_{\vec{0}}$ be the soln set of $C\vec{x} = \vec{0}$, let $S_{\vec{b}}$ be the soln set of $C\vec{x} = \vec{b}$, and let $\vec{v}$ be a particular soln of $C\vec{x} = \vec{b}$.

Then we have $S_{\vec{b}} = S_{\vec{0}} + \vec{v}$, i.e.,

$$S_{\vec{b}} = \{ \vec{w} + \vec{v} \mid \vec{w} \in S_{\vec{0}} \}.$$

So geometrically $S_{\vec{b}}$ is a shift of $S_{\vec{0}}$.

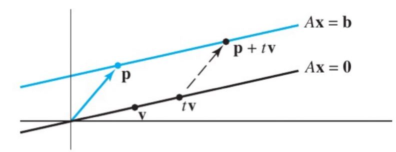


FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

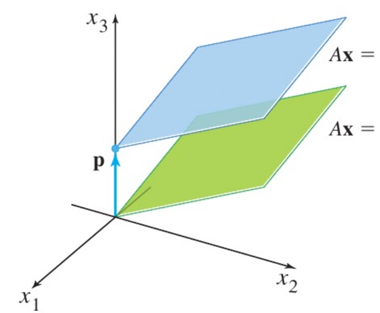


FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Ex. Work out the soln sets of $\begin{cases} x+2y-z=4 \\ x-y+z=0 \end{cases}$

and $\begin{cases} x+2y-z=0 \\ x-y+z=0 \end{cases}$. Draw them, compare them and note that they fit the descriptions of the theorem.

Next time:

linear transformations